Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Global versus local symmetries

Master's Thesis

2023

Bc. Ján Pastorek

Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Global versus local symmetries

Master's Thesis

Study Programme:	Cognitive Science
Field of Study:	Computer Science
Department:	Department of Applied Informatics
Supervisor:	doc. RNDr. Tatiana Jajcayová, PhD.

Bratislava, 2023

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THESIS ASSIGNMENT

Name and Surname:	Bc. Ján Pastorek
Study programme:	Cognitive Science (Single degree study, master II. deg., full time form)
Field of Study:	Computer Science
Type of Thesis:	Diploma Thesis
Language of Thesis:	English
Secondary language:	Slovak

Title: Global versus local symmetries

Annotation: To study an object, whether natural or abstract, an important part of it is the understanding of its symmetries. The sciences developed various tools to describe symmetries, in particular in mathematics, the classical tools are group theoretic tools to study permutations. It become apparent that the total symmetries are very restrictive and not giving the full information about the studied object, as it can have no full perfect global symmetry but still be "somewhat symmetric". Recently, the research is done to capture this relaxed notion of symmetry, the partial (or local) symmetry, and develop formal tools to describe it.

Aim: The goal of the project is to analyze how the fundamental concept of partial (local) symmetry is treated in various different areas: Philosophy & history, Cognitive psychology, Aesthetics & Arts, Nature & Natural Sciences, and Mathematics & Computer Science. We want to find and emphasize the unifying principles in these different approaches to the symmetry. In the thesis the argument will be given that the concept of local (partial) as opposed to global (total) symmetry is more natural, more general and better describes the natural phenomena as well as symmetries in abstract structures. Also, the formal mathematical tools ranging from the classical group theoretical tools for studying total symmetries to very resent generalizations to inverse semigroup theory to handle partial symmetries will be used in some creative ways to illustrate their usefulness.

Literature:
Borrelli, A. (2019). Between symmetry and asymmetry: Spontaneous symmetry breaking as narrative knowing. Synthese, 198(4), 3919–3948.
Jajcay, R., Jajcayová, T., Szakács, N., & Szendrei, M. B. (2021). Inverse monoids of partial graph automorphisms. Journal of Algebraic Combinatorics, 53(3), 829–849.
Penrose, R. (1974). The role of aesthetics in pure and applied mathematical research. Bull. Inst. Math. Appl., 10, 266–271.
Lawson, M. V. (1998). Inverse semigroups, the theory of partial symmetries. World Scientific.
Shubnikov, A. V., & Koptsik, V. Aleksandrovich. (1974). Symmetry in Science and Art.

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ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta: Bc. Ján Pastorek
Študijný program: kognitívna veda (Jednoodborové štúdium, magisterský II. st., denná forma)
Študijný odbor: informatika
Typ záverečnej práce: diplomová
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Sekundárny jazyk: slovenský

Názov: Global versus local symmetries
Globálne verzus lokálne symetrie

Anotácia: Keď skúmame nejaký objekt, či už prírodný alebo abstraktný, je dôležité porozumieť jeho symetriám. Vedné disciplíny vyvinuli rôzne nástroje na opis symetrií. Špeciálne v matematike, klasické nástroje sú nástroje teórie grúp na štúdium permutácií. Ukazuje sa, že úplné globálne symetrie sú veľmi obmedzujúce a neposkytujú celú informáciu o študovanom objekte, pretože ten môže nemať žiadnu dokonalú globálnu symetriu a je teda asymetrický, ale stále môže byť „trochu symetrický". V súčasnosti prebieha veľmi živý výskum s cieľom zachytiť tento zovšeobecnený koncept symetrie, nazývanej čiastočná alebo lokálna, a vyvinúť formálne nástroje na jej opis.

Cieľ: Cieľom projektu je analyzovať, ako sa uvažuje o základnom koncepte čiastočnej (lokálnej) symetrie v rôznych oblastiach ľudského bádania: filozofia a história, kognitívna psychológia, estetika a umenie, príroda a prírodné vedy a matematika a informatika. Chceme nájsť a zdôrazniť zjednocujúce princípy v týchto rôznych prístupoch k symetrii. V práci sa bude argumentovať, že koncept lokálnej (čiastočnej) na rozdiel od globálnej (totálnej) symetrie je prirodzenejší, všeobecnejší a lepšie vystihuje prírodné javy ako aj symetrie v abstraktných štruktúrach. Cieľom bude tiež používať formálne matematické nástroje od klasických z teórie grúp na štúdium globálnych symetrií až po veľmi moderné zovšeobecnenia do inverzných pologrúp na popis čiastočných symetrií, na niektoré problémy napr. v teorii grafov na ilustráciu užitočnosti týchto nástrojov.

Literatúra:

Borrelli, A. (2019). Between symmetry and asymmetry: Spontaneous symmetry breaking as narrative knowing. Synthese, 198(4), 3919–3948.
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Dátum zadania: 09.03.2023

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študent

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vedúci práce

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Abstract

Symmetry has been one of the foundational principles of human thought and aesthetics since ancient times, with its roots tracing back to various civilizations that appreciated balance, harmony, and order in their art, architecture, and philosophy. The term 'symmetry' is derived from the Greek words 'syn' (together) and 'metron' (measure), reflecting among others the idea of equal arrangement and proportion (Hon & Goldstein, 2008). However, modern understanding of 'global' symmetry is reduced to the set of transformations that leave the object invariant, as understood by mathematical group theory.

In this thesis, we investigate the concept of 'local' (partial) symmetry, which may be looked at as a sort of return to the original meaning of the term of symmetry, stressing the importance of a proportionality but capturing the current meaning of global symmetry as well. Moreover, we investigate its significance in various disciplines, including Philosophy, Arts, and Natural sciences.

Furthermore, we argue that the concept of local (partial) symmetry, as opposed to global (total) symmetry, is more natural, more general, and better describes natural phenomena and symmetries in abstract structures. The classical concept of global symmetry is insufficient to capture the complexity of the world because it is very restrictive to slight asymmetries, and symmetries hold too much redundant information and so are extremely simple to generate complexity, while the concept of partial symmetry provides a more accurate representation. Since the concept of partial symmetry focuses on proportion, i.e., both on the whole and the parts, therefore it captures what master tilers do when they tile objects.

In order to build a comprehensive and consistent understanding of partial symmetry, a rigorous and systematic approach is employed, utilizing algebraic methods from the field of mathematics. Following Jajcay et al. (2021) and Jajcayova (2022) and Lawson (1998), the relevant part of the theory of inverse monoids is developed through the careful formulation of hypotheses and logical reasoning. Furthermore, perspectives from various disciplines are synthesized to analyze how partial symmetry is treated in different areas and to identify unifying principles in these approaches. Moreover, we used many examples and illustrations in order to synthesize a comprehensive and consistent understanding of the concept of (partial) symmetry.

Keywords: Group, Inverse monoid, Symmetry, Complexity, Structure, Fractal, Information, Proportion

Abstrakt

Symetria je jedným zo základných princípov ľudského myslenia a estetiky už od staroveku, pričom jej korene siahajú k rôznym civilizáciám, ktoré oceňovali rovnováhu, harmóniu a poriadok vo svojom umení, architektúre a filozofii. Pojem 'symetria' je odvodený z gréckych slov 'syn' (spolu) a 'metron' (miera), odrážajúc okrem iného myšlienku rovnakého usporiadania a pomeru (Hon & Goldstein, 2008). Moderné chápanie 'globálnej' symetrie sa však redukuje len na množinu transformácií, ktoré ponechávajú objekt nemenný, v rámci matematickej teórie grúp.

V našej práci skúmame koncept 'lokálnej' (čiastočnej) symetrie, ktorý možno považovať za akýsi návrat k pôvodnému významu pojmu symetria, zdôrazňujúc dôležitosť proporcionality, ale zároveň zachytávajúci súčasný význam symetrie. Okrem toho skúmame jeho význam v rôznych disciplínach vrátane filozofie, umenia a prírodných vied.

Tvrdíme, že koncept lokálnej (čiastočnej) symetrie je na rozdiel od globálnej (úplnej) symetrie prirodzenejší, všeobecnejší a lepšie vystihuje prírodné javy a symetrie v abstraktných štruktúrach. Klasický koncept globálnej symetrie nestačí na zachytenie zložitosti sveta, pretože je veľmi reštriktívny, čo i len na nepatrné asymetrie. Tiež, symetrie obsahujú príliš veľa nadbytočných informácií, a preto sú extrémne jednoduché na vytváranie zložitosti, zatiaľ čo koncept čiastočnej symetrie poskytuje presnejšie zobrazenie Keďže koncept čiastočnej symetrie sa zameriava na proporcie, t.j. ako na celok, tak aj časti, preto tento koncept zachytáva, čo robia majstri umelci, keď obkladajú plochy.

Na vybudovanie komplexného a konzistentného chápania parciálnej symetrie sa v tejto práci používa rigorózny a systematický prístup využívajúci algebraické metódy z oblasti matematiky, prebraný z výskumu matematikov. Na základe vedeckých článkov Jajcay et al. (2021) a Jajcayova (2022) a knihy Lawson (1998), rozvíjame príslušnú časť teórie inverzných monoidov prostredníctvom formulovania hypotéz a logického uvažovania. Okrem toho sme zosyntetizovali pohľady z rôznych disciplín, ako sa v týchto rôznych oblastiach zaobchádza s čiastočnou symetriou, aby sme identifikovali zjednocujúce princípy v týchto prístupoch. Navyše použili sme veľa príkladov a ilustrácií, aby sme vytvorili komplexné a konzistentné pochopenie pojmu (čiastočná) lokálna symetria.

Kľúčové slová: Grupa, Inverzný monoid, Symetria, Zložitosť, Štruktúra, Fraktál, Informácia, Proporcia

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Introduction

Mirrored steps in chains of order, the colors fade,
But vibrant dance, proportions breathe, persuade.
In chaos' sweet embrace, we find our fire,
As imperfections dance and hearts aspire.

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I have two aims in mind for this thesis. The first is to demonstrate the wide range of applications of the principle of partial symmetry in a variety of fields, such as the arts, and nature, and contrast it to the usage of global symmetry. The second aim is to explain the philosophical and mathematical significance of the idea of partial symmetry as opposed to the concept of global symmetry in a clear and gradual manner.

Apart from others, these questions are addressed: Is the concept of partial symmetry more fundamental than global symmetry? In what sense? How does (partial) symmetry influence perceptual organization, attention, and thought? How does it relate to other concepts such as complexity?

The main thesis is that the concept of local (partial) as opposed to global (total) symmetry is more natural, more general, and better describes the natural phenomena as well as symmetries in abstract structures. Note that we use the terms 'local' and 'partial' symmetry interchangeably. Similarly, we use the terms 'total', 'global', and 'perfect' symmetry interchangeably. Formal mathematical tools ranging from the classical group theoretical tools for studying total symmetries to very recent generalizations to inverse monoid theory to handle partial symmetries will be used in some creative ways to illustrate their usefulness. Moreover, I use many examples and images in order to illustrate the argument.

The formalism and roughly the general idea of the argument for the more generalized concept of symmetries that I present is known at least since the failure of the initial Klein's "Erlanger program" in the late 19th century to characterize all kinds of geometries based on the groups of symmetries (Lawson, 1998). Nevertheless, this thesis aspires to be the first modern interdisciplinary, systematic, and integrative (and hopefully consistent) exploration of the concept of partial symmetries, and in this sense similar to Weyl's exploration of the concept of global symmetry in 1989. And, to the best of

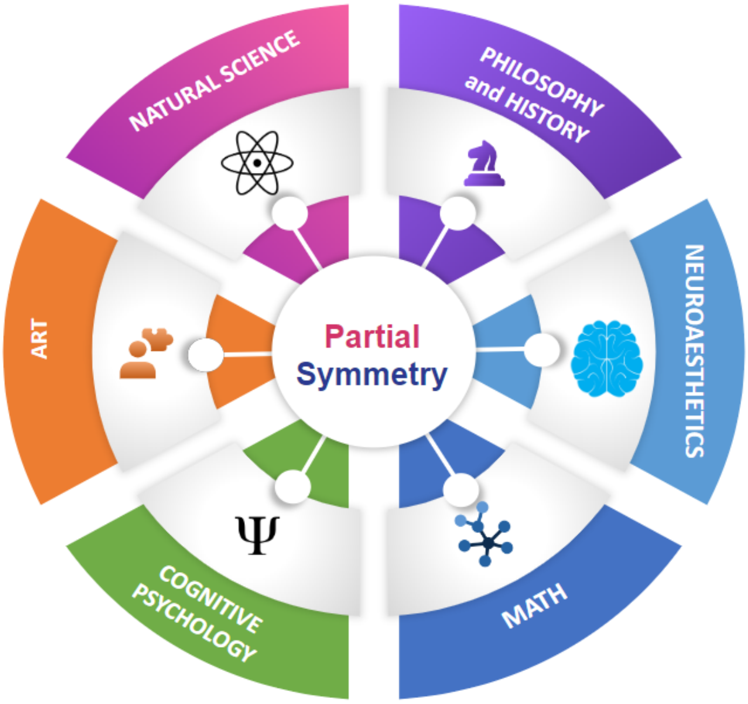

Figure 1: Interdisciplinary perspectives on the concept of (partial) symmetry

my knowledge, it is the first such exploration.

I intend for this thesis to be accessible to a wider audience than just specialists in specific fields. And, this is my biggest challenge, to be both precise and stay accessible. Thus, while the thesis does not avoid using mathematics, it is not intended to provide a detailed mathematical treatment of all the problems it covers. Such an approach would be outside the thesis' scope and defeat its purpose.

The structure of the thesis follows the same two paths that we describe in the description of the history of the concept of symmetry in Section 0.1: phenomenological/aesthetic in Chapter 1 and Chapter 2 and mathematical/formal in Chapter 3. We choose this division because it follows the historical evolution of the concept and because in phenomenology and aesthetics, the concept of partial symmetry may not always be as precise as in the formal approach. Moreover, that is how we tend to approach the world around us – from examples to abstractions.

I synthesize these two paths in Chapter 4 which deals with the whole argument for the concept of partial symmetry, and therefore cross-references to other chapters where the particularities are described. Moreover, we treat the concept of (partial) symmetry in various areas: Philosophy & History, Cognitive psychology & Neuroaesthetics, Aesthetics & Arts, Nature & Natural Sciences, and Mathematics & Computer Science (illustratively shown in Fig. 1). Moreover, I want to find and emphasize the unifying principles in these different approaches to partial symmetry.

We arranged the thesis in natural order from particularities to abstraction. However, there are some exceptional sections, such as in Chapter 4 which cross-reference a lot to other chapters for a more detailed understanding of the argument. Thus, at times, the reader may find it more useful to jump back into these sections for more details. The reader with an understanding of abstract algebra may omit at least the initial definitions and proofs in Chapter 3.

0.1 Evolution of the concept of (partial) symmetry

The key question that we are trying to explore in this section is how the meaning of the concept of (partial) symmetry has evolved over time. Hon and Goldstein (2008) challenged the received view that symmetry is a concept that has always been available to human thought in its modern meaning. Their research reveals that the term ***symmetry*** did not carry its modern meanings until the 18th century. They identify two different trajectories for the evolution of the meaning of the term symmetry: the mathematical path and the aesthetic path. In this section, we summarize the results of historical analysis, mostly based on reviews of Brading et al. (2003), Hon and Goldstein (2008), and Nagy (2022). For more details, see these references. We also roughly outline the evolution of the term in Fig. 2. As you will see, the concept gradually lost its previous meanings for the sake of being formalizable from a rather vague notion that relates to many ideas such as proportion, harmony, and commensurability, into a mathematically precise explicit concept as defined by group theory emphasizing the ideas of inverse order and relation as opposed to the ideas of a whole, proportionality, and property.

0.1.1 Aesthetic path

In the realm of aesthetics, Plato and his pupil Aristotle considered a living creature to be beautiful if it was symmetrical, or well-proportioned in between the 5th and 4th century BC. For instance, in his work *Poetics* Aristotle says that "to be beautiful, a living creature, and every whole made up of parts, must . . . [be] present a certain order in its arrangement of parts." Aristotle also identified order, symmetry, and definiteness as chief forms of beauty. In his work *Metaphysics*, he states, "The chief forms of beauty are order and symmetry and definiteness, which the mathematical sciences demonstrate in a special degree" (Aristotle, 2016).

In the realm of morality, in his work *Laws*, Plato used the term ***summetros*** to mean also ***suitable*** or ***appropriate*** (Plato, 1962). Aristotle also used the term in moral context. For him, suitability or moderation was considered the guiding principle for good conduct. Recall that his heuristic towards leading a good life was golden mean, the principle roughly saying that to lead a good life you should avoid the extreme states of excess and deficiency. For instance, avoid foolhardiness, and also avoid cowardice.

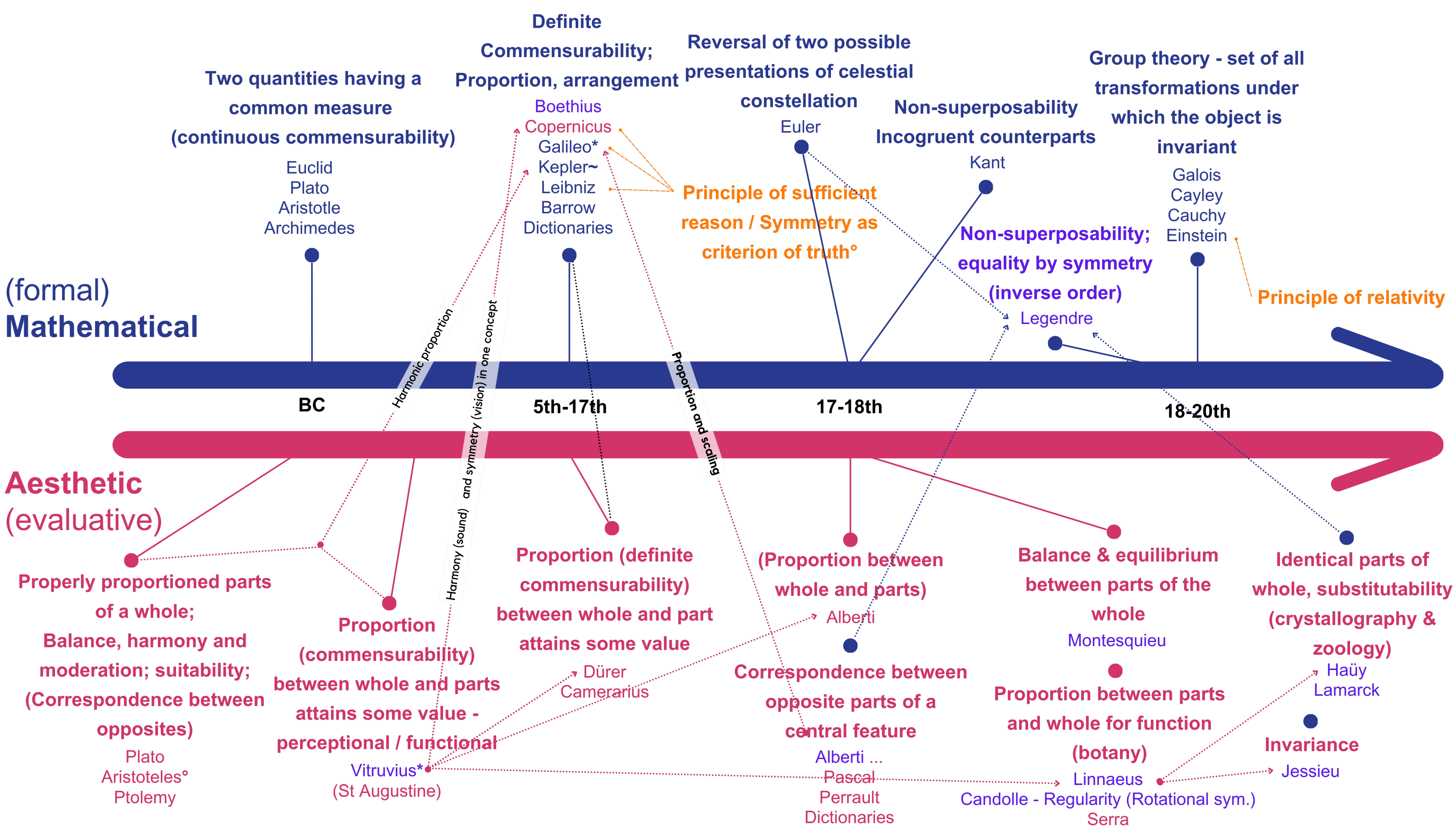

Figure 2: The evolution of the meaning of the term *symmetry*. Following Hon and Goldstein (2008) two paths of evolution of the term are displayed. Note that there were many influences that crossed this simple differentiation between the paths, some indicated by cross-relation or color in the picture. Only, central ideas and key thinkers and artists are shown that influenced the transitions and meaning of this term. People and ideas that are in purple are especially important in the transitions of meaning towards the modern meaning of the term.

Moreover, Ptolemy referred to symmetry in the *Almagest* as ***suitable in size***.

Later in the 1st century BC, Vitruvius, a Roman architect, employed symmetry in the ancient sense in the design of buildings and machines. For Vitruvius, symmetry was an attribute of a whole object, such as a human body, building, or machine, when the parts were joined in a way that formed a beautiful and well-coordinated entity. Virtuvius added to the meaning a certain subjective flexible evaluation since the proportionality was oriented towards attaining some perceptional or functional value. As we will see, Vitruvius's work *De architectura* had a significant impact on European thought and practice, shaping studies in art and engineering for almost two millennia.

Albrecht Dürer (1471–1528), a German artist, extended and elaborated on Vitruvius's concept of symmetry in his work *Vier Bücher von menschlicher Proportion*, which especially focused on human proportion. Otherwise, early modern times saw the acceptance of Vitruvius's concept of symmetry with little change.

It was until 15th century that the meaning stayed relatively the same in aesthetic path. Symmetry played a central role in architecture, linking harmony and beauty through specific proportions for parts of a structure in relation to each other and the whole. The significance of symmetry in architecture is highlighted in Renaissance architect Leon Battista Alberti's (1404–1472) own words: "Beauty is that reasoned harmony of all the parts within a body, so that nothing may be added, taken away, or altered, but for the worse." In this way, Alberti moved the meaning of the term ***symmetry*** towards ***harmony of parts and whole***, despite the fact that the word ***harmony*** was only used in the domain of music at that time. This approach supported the use of ***scale modeling*** in architecture, with the idea that what works on a small scale will also work on a large scale.

However, Hon and Goldstein (2008) argue that the term ***correspondence*** facilitated a shift away from Vitruvian concept of symmetry, as the Alberti introduced it as a methodological principle. Although Aristotle and Augustine discussed the arrangement of identical parts in living creatures, they did not use the term ***correspondence*** in the same way as Alberti. In Alberti, "correspondence is not a relation between the part and the whole; rather, it is a relation between one part and another part of a single edifice, and the relation is fixed, namely, the two parts are identical" (Hon & Goldstein, 2008). Thus, he distinguished the concept of symmetry as ***proportion*** from the idea of ***correspondence***. Nevertheless, Alberti still recognized the original Vitruvian meaning of the term but chose to introduce the ***correspondence*** into the meaning, differentiating between the two meanings. Thus, Alberti appealed to direct observation of nature and ancient artifacts, emphasizing the relationship between correspondent parts of an edifice rather than the part-whole relation.

Then between 1673 and 1683, in the French tradition, Perrault explicitly distinguished the French usage of ***symmetrie*** from the ancient usage of ***symmetria*** to avoid ambiguity. He defined symmetry as a relationship of ***parity and equality of left and right***, which was distinct from the classical meanings. Hon and Goldstein (2008) argue that this definition and distinction represented a crucial turning point in the emergence of the modern concept of symmetry.

In 1757, Montesquieu combined two opposing sources of pleasure — order and variety/chaos — to argue that a pleasing object must possess both simplicity and diversity in a balance. He conceptualized symmetry as an ordering principle or balance that followed Perrault's concept, rather than Vitruvian symmetry. As opposed to Perrault's use of parity and equality of left and right, Montesquieu stressed the ***balance between the two opposing sides***. Interestingly, the proportion, which is at the root of Vitruvian symmetry, was not even mentioned in Montesquieu's work, indicating a gradual shift from the meaning of symmetry as proportion.

Moreover, in 1757, Burke criticized the theory of proportion as the source of beauty in art and architecture, arguing that mathematical ideas are not the true measures of beauty. He claimed that proportion in art was derived from transferring artificial ideas to nature, rather than borrowing these ideas from nature. As we can observe, symmetry as ***proportion*** gradually shifted from this meaning of the term, which was heavily opposed in the work of Burke.

Carl Linnaeus (1707–1778), considered one of the fathers of modern biology, introduced a system of classification and nomenclature in the field. Returning to the previous meaning of the word, he recognized plant symmetry as a whole resulting from the relative disposition of interacting parts, which were not necessarily mathematical or aesthetically based. This concept of symmetry was built upon Vitruvius' discussion of symmetry in machines, where symmetry referred to the proper functioning of parts working smoothly together.

In 1777, the concept of ***invariability*** was introduced in natural history and later linked to symmetry. Antoine Laurent de Jussieu expanded Linnaeus's system by adding secondary characters and seeking invariability in essential plant parts. Although botanists of the time did not see a formal connection between symmetry and invariability, the idea was there to be explored further.

De Candolle (1778–1841) further developed Linnaeus's principle of symmetry and conducted a subtle analysis of plant structures based on relative disposition and the tendency of plants toward symmetrical wholes. He defined symmetry as the particular arrangement of parts resulting from their relative position and forms, such as the insertion of stamens, the structure of the fruit, and the organization of seed parts.

Haüy (1743–1822) made a significant move toward the modern meaning of rotational and bilateral symmetry in mineralogy and crystallography. He sought an analog to botanical regularities of form and considered symmetry a geometrical property of a single object. Haüy's concept of symmetry expressed a kind of rotational symmetry, and the term *axis* was introduced as an imaginary line linking diametrically opposite parts of a crystal's surface. While viewing Haüy's definition from a modern perspective, might suggest that he was the first one to actually capture the modern meaning, however, Haüy was not explicit and abstract enough about his concept of symmetry and did not use the concept of axis in the context of symmetry. Despite this, these ideas of rotation are incorporated into the modern concept of symmetry.

Lamarck (1744–1829) also invoked the term symmetry in his discussion of worms, stating that their general form consisted of a symmetrical opposition in its parts, with each part being completely similar to the opposite one.

0.1.2 Mathematical path

In the mathematical path, the meaning of symmetry remained stable for centuries, but eventually fell out of active use. Initially, in the 4th and 3rd century BC, Euclid and Archimedes, and other mathematicians and philosophers of antiquity used the terms *summetra* and *asummetra* in Greek. At that time, the mathematical sense of symmetry roughly focused on commensurability, i.e., two quantities are commensurable if they have a common measure, and so the terms were later translated into Latin as *commensurabiles* and *incommensurabiles* by Boethius between the 5th and 6th century AD. However, Euclid initially meant by commensurability even continuous commensurability, i.e., two quantities can have a common measure even if this common measure is a rational number, later, only integers were understood as common measures.

Copernicus (1473–1543) believed that God, as the most orderly Artisan, would create a universe with compatible elements, thus, he was appealing to some underlying assumption of the orderliness of nature. He also alluded to three aspects of symmetry as well-proportioned feature of an object, proportioned structure and proper functioning of a machine that were mentioned by Vitruvius separately. Copernicus applied them in new ways while still retaining the same meaning. He linked these aspects of symmetry and even harmony (before used only in the consideration of sound) on a cosmic scale, arguing that the parts of the cosmos fit together to form a well-proportioned harmony, a perfect functioning and structured whole. This combination of concepts was likely Copernicus' own creativity, as there is no evidence of these ideas being linked in any source available to him.

Further on, Galileo (1564–1642) touched upon the concept of scaling (the idea that what works on a small scale will also work on a large scale), which was not found in ancient literature, perhaps only in Alberti's own work. He also used symmetry as a scientific *criterion of truth*, stating that a system without symmetry must be faulty. For instance, he argued against the Ptolemaic system, which did not exhibit symmetry, and therefore could not be a true description of the world. However, Galileo only referred to symmetry, not harmony, unlike Copernicus.

Kepler (1571–1630), on the other hand, did not appeal to symmetry in his work. Instead, he focused on ***cosmic harmony***, ignoring Copernicus's invocation of symmetry. He believed that "the archetype of the movable world was constituted not only by the five [Platonic] regular [solid] figures... but also by the harmonic proportions with which the courses themselves were attuned." Moreover, Kepler paid enormous attention to the idea of ***divine proportion / golden ratio***.

Leibniz (1646–1716), like Kepler, chose to place ***harmony*** at the foundation of his thinking. His concept of universal harmony is considered one of the most central features of his thought. Harmony, in Leibniz's view, encompasses both ***unity and multiplicity***. To the best of our knowledge, what he meant by these terms is yet unclear. However, he did not use the term symmetry. Instead, he saw harmony as connected with order, regularity, and uniformity. This is demonstrated in one of his remarks: "Rules are the expression of general will: the more one observes rules, the more regularity there is; simplicity and productivity are the aim of rules. I shall be met with the objection that a uniform system will be free from irregularities. I answer that it would be an irregularity to be too uniform, that would offend against the rules of harmony."

Moreover, Leibniz's metaphysical system was built upon the idea that the universe exhibits a pre-established harmony, which allows for a coordinated functioning of all its parts. This harmony is essential for the existence of the world and serves as the foundation for the interconnectedness and order found within the universe. Leibniz believed that God, as the perfect creator, had designed the world in such a way that it exhibits this harmonious balance between unity and multiplicity, thus, he was appealing to pre-established order as Copernicus. We can conclude that his idea of harmony is a sort of ***balance between order and chaos***.

While Copernicus and Galileo used the term in a manner consistent with the Vitruvian tradition, Leibniz chose not to link symmetry to harmony. His focus on harmony, rather than symmetry, highlights the importance he placed on the balance between unity and multiplicity in the universe, reflecting his belief that rather than symmetry in whichever sense, harmony is essential for a well-ordered and interconnected world.

Later on, in 1750, Euler, a renowned mathematician, received an anonymous question regarding the preference of drawing a star constellation on a celestial globe, which was a huge rotating imitation of the seen star constellation on the globe. The query he received was this:

> One imagines the constellations on the inner surface of the sky either as real bodies or as bodies that were painted with transparent colors. They are painted on the globe from the outside, quite the way they would appear

> to us if we could look at the starry sky from the outside at a great distance. It is clear that, with this position of the eye, we would see the constellations from behind [von hinten], if they were real bodies. [On the other hand], we would see them from the front [von vorne], but ***reversed*** [verkehrt], if they were drawn with transparent colors. Which of the two is to be preferred, and which is the most convenient [bequemste] for forming a conception of the location of the stars and for recognizing those [constellations] that were described for us by the ancients? Deciding this question will be left to Mr. Euler (Hon & Goldstein, 2008)

The meaning of the question is roughly this: Imagine you have a transparent ball (the celestial globe) with stars and constellations painted on its surface. There are two ways to paint the constellations:

The first, paint the constellations on the inside of the ball, treating them as real 3D objects seen from the earth. If you were to look at the ball from the outside, the constellations would appear as if you were looking at the back of 3D objects.

For the second, paint the constellations on the outside of the ball, treating them as 2D drawings in front of you. If you were to look at the ball from the outside, the constellations would appear as if you were looking at the front of 2D drawings, but they would appear reversed (mirrored) from the inside of the globe.

While Euler spotted the equivalence of these two mirror solutions and responded to this query as such saying that from a mathematical perspective, the distinction between the left and the right is indifferent, nevertheless, he did not generalize this problem, nor spot its potential importance for geometry.

Further, in 1768, Kant explored the concept of ***incongruent counterparts***, such as left and right hands, as a means to understand the nature of space. He observed how these counterparts could be equal in magnitude and equal in form, yet non-superposable due to their differing directionality, where ***superposability*** means the possibility of one object being transported onto another so that they coincide. He was trying to use this concept to demonstrate the real existence of the quality of directionality of absolute space. Today, we call such non-superposable objects, chiral.

However, Kant's initial attempt to use mathematics to capture the essence of space was unsuccessful, leading him to abandon the concept of ***incongruent counterparts*** in favor of a metaphysical analysis of space as pure intuition (Kant, 2004). This shift is evident in his *Prolegomena* from 1783.

In retrospect, Kant's approach involved abstracting objects and considering their form, regardless of the specific object in question, in order to study the problem of form and its directionality in space. He recognized that all these objects posed the same problem of non-superposability.

Moreover, Kant's concept of incongruent counterparts transcended the aesthetic component and was not limited to human bodies or architectural structures, unlike the French tradition of usages of symmetry. His focus was on the mutual relation of objects in space, rather than the relation between two halves of a single object, as in the French usage of the concept.

Kant aimed to establish an inner spatial property by defining an absolute distinction, such as an object being a right-hand in reference to a universal absolute space. However, when he could not establish this, he abandoned this approach in favor of a metaphysical one, where space could be apprehended via intuition (Hon & Goldstein, 2008).

Hon and Goldstein (2008) argue that Kant and Euler were actually the closest toward the revolutionary modern meaning of symmetry since they both understood mirror-image. However, it was a mathematician Adrien-Marie Legendre who in 1794 introduced a radical new mathematical meaning for the term symmetry, defining it as a *relation* rather than a property of an object. Legendre, like other mathematicians in the 18th century, was interested in solid geometry, and mathematicians at that time had to address ways to change Euclidean definitions of similarity and equality to work properly in a three-dimensional framework. Legendre overcame the difficulties in Euclid's formulations and introduced symmetry into geometry as a relation of equality/congruence between two solids that are non-superposable, thus he reintroduced it into Euclidean geometry as a precise mathematical concept that differs radically from its previous usage by Euclid. He defines symmetrical solid angles as equal solid angles formed by the same plane angles but in ***inverse order***.

Legendre's innovative definition of symmetry demonstrates that his new concept is unrelated to the proportionality of parts with respect to a whole, which is the defining characteristic of the ancient concept of symmetry. The innovativeness in his definition that shifts it close to the modern meaning can be roughly summarized:

1. The term symmetry is mathematically precisely formulated in the geometrical sense of solid geometry, thus, Legendre's definition of symmetry is categorical and explicit.

2. "Symmetry becomes a relation between two solid figures, irrespective of their arrangement in space," and not just a property of a single object.

3. He recognizes the significance of inverting the order of angles that nevertheless preserve equality.

4. He drops the relation of parts to a whole. Brading (2010) thinks that this relationship is actually part of the modern concept, and so he does not agree on this with Hon and Goldstein (2008). As we will see in Chapter 3, this is true to a

> certain degree – the modern concept captures only a global perspective on the parts, not the local perspective of parts to the whole.

Therefore, his definition significantly differs from the used meaning of symmetry in architecture, where the ***whole*** plays an essential role and the parts must be arranged in some manner. Moreover, this novel concept of symmetry became a powerful concept in scientific domains, proving to be fruitful for subsequent work in many scientific disciplines, as shown by a subsequent growing usage of the term in the work of Cauchy (1813) and other mathematicians, and applications in physics.

Later, the concept was further generalized into the modern ***group theory*** which was developed by mathematician Galois in the early 19th century, but the central meaning remained roughly the same. In its modern sense, it is usually stated that symmetry denotes the ***set of all transformations that leave the object invariant***/the same. However, the modern concept encompasses not only usual or geometrical objects but is also frequently applied to laws and equations. If you compare it to Legendre's definition, it is a much broader concept, since it also encompasses rotational symmetry and identity/doing nothing. We will walk through this modern formalization in Chapter 3, and as the reader will see, the inverse order is essential for the modern concept.

There were multiple cross-talks between the two evolutionary trajectories of the concept of symmetry as indicated in the Fig. 2. Galileo and Copernicus were inspired by the concept of harmony which they transferred into the mathematics of the cosmos. Perhaps the most significant cross-talk was the contribution of Alberti who distinguished and separated the meaning of symmetry in terms of proportionality from the meaning of correspondence between opposites, since it allowed subsequent separate development of the two distinguished concepts.

In summary, research of Hon and Goldstein (2008) shows that the concept of symmetry has evolved significantly over time, with partially distinct mathematical and aesthetic trajectories shaping its development. The modern understanding of symmetry as a scientific concept emerged only in the 18th century in the work of Legendre. Before the 18th century, there is no evidence that the modern concept of symmetry was used.

Moreover, I will show that we can look at the concept of partial (local) symmetry as a sort of return to the original meaning of the term symmetry, stressing the importance of proportionality and locality while capturing the modern meaning of symmetry as well. I will demonstrate this in Section 3.3.

0.2 Symmetry and symmetry breaking

In physics, symmetry principles have played a crucial role in our understanding of the fundamental forces and particles in the universe. For example, the laws of physics are symmetric under translations in space and time, known as Noether's laws (Brading et al., 2003; Noether, 1971). The fundamental goal of physics is to unify these laws using a single superprinciple of symmetry (Mainzer, 2005).

theory reduction ↑ / symmetry breaking ↓

theories	models	symmetries
string theories theory of relativity quantum field theories	strings, *p*-branes gravitons, photons hadrons, leptons, etc.	supersymmetry space-time symmetries dynamical symmetries (e.g., SU(3)xSU(2)xU(1)-forces)
chemistry biochemistry thermodynamics	atoms, molecules, etc. proteins, DNA, etc. open systems with metabolism	nuclear-, orbital-, crystal symmetries chirality dissipative structures of symmetry
biology ecology economics	organisms populations economics, markets	functional symmetries ecological balance economic balance, economic equilibrium
sociology	societies	social balance, social equilibrium

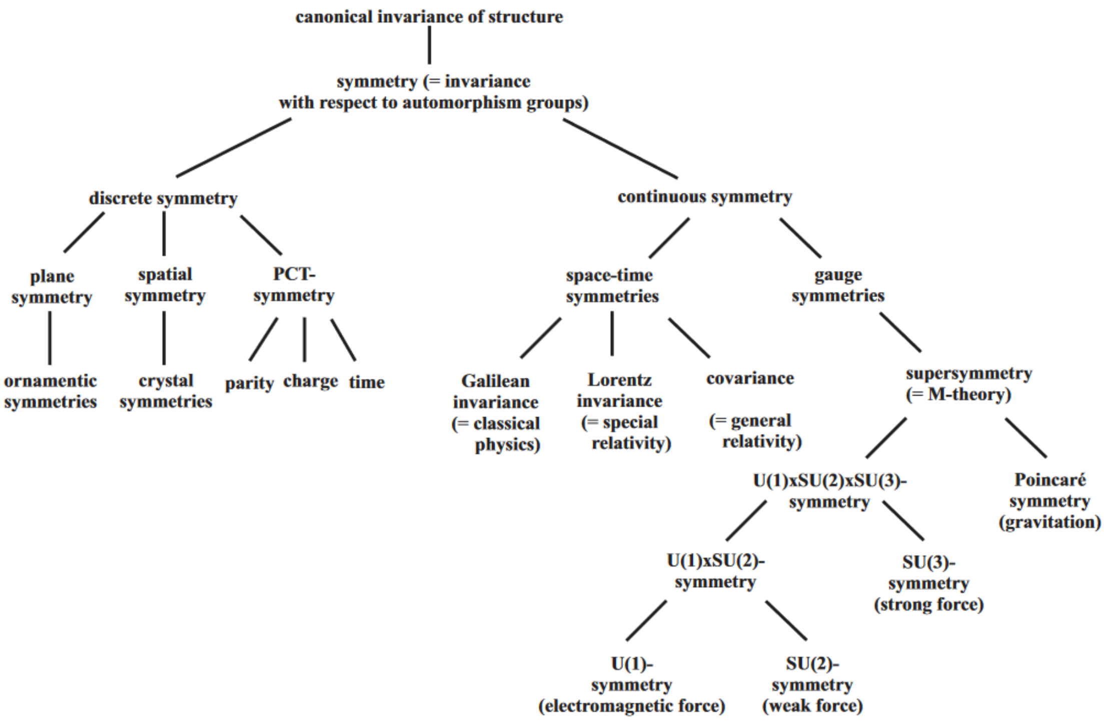

Figure 3: Narrative of symmetry breaking with corresponding ontology of underlying symmetries that is supposed to generate all complexity. After Mainzer (2005)

However, various types of *symmetry breaking* have been observed in physical systems. These types can be classified in the following way:

Spontaneous Symmetry Breaking: This occurs when a system moves from a symmetric state to a lower-energy, asymmetric state without any external influence. A well-known example is the Higgs mechanism, which explains the origin of mass for fundamental particles (Brading et al., 2003). According to this mechanism, there is a field called the Higgs field that permeates all of space. As particles move through this field, they ‘experience’ resistance, much like moving through a viscous fluid. This resistance results in particles acquiring mass. The Higgs field is associated with a particle called the Higgs boson, which was discovered in 2012. As such, the concept of spontaneous symmetry breaking is a hybrid narrative explaining how asymmetry arises from symmetry without the symmetry being lost, but only somehow hidden at a more fundamental level of reality. Although it might be practical to start from an asymmetric model and never consider hidden symmetries, doing so would eliminate the explanatory potential displayed by spontaneous symmetry breaking in current scientific practice and that is why physicists rather rely on the story of hidden deeper symmetries (Borrelli, 2019).

Explicit Symmetry Breaking: This happens when a system’s symmetry is broken due to external factors, such as the weak nuclear force’s violation of parity symmetry in particle interactions. Parity symmetry was the prediction that the particles’ nucleus behaves in the same way if its spatial configuration is reversed. We now know that this is not true in all cases (Aubrecht, 2003; Brading et al., 2003).

From a philosophical perspective, symmetry breaking raises questions about the nature of symmetry itself. If perfect symmetry exists in nature, why do systems and laws frequently exhibit symmetry breaking? The common narrative is that symmetry breaking just shows a need to look for deeper symmetries, and is related to phase transitions which lead to emergence (Borrelli, 2019; Mainzer, 1996, 2005) This view that symmetry and symmetry breaking can explain all the phenomena is prevalent in modern science, especially in physics, and can be roughly illustrated as in Fig. 3.

However, we argue that the very idea of perfect symmetry might be an idealization that does not have to exist in nature. Instead, imperfection and slight asymmetry might be inherent properties of the universe, with symmetry breaking serving as an indication of this fundamental aspect.

Moreover, symmetry breaking is often linked to the emergence of complex structures and patterns in nature, such as the formation of crystals, the development of organisms, or the evolution of galaxies (Mainzer, 1996, 2005). This suggests that the lack of perfect symmetry is necessary for the richness and diversity we observe in the natural world. We will further explore this different narrative.

0.3 Partial symmetry and nature

It was Schrödinger who first pointed out that the structure of living matter is much more complex beyond anything studied in physics and chemists at that time. He thought that living matter is ***non-periodic*** (its structure is not like a lattice that repeats the same pattern again and again in all directions), and thus, not easily describable by the laws that describe classical physics. In his own words,

> structure of the vital parts of living organisms differs so entirely from that of any piece of matter that we physicists and chemists have ever handled physically in our laboratories or mentally at our writing desks. It is [almost] unthinkable that the laws and regularities thus discovered should happen to apply immediately to the behaviour of systems which do not exhibit the structure on which those laws and regularities are based. The non-physicist cannot be expected even to grasp let alone to appreciate the relevance of the difference in 'statistical structure' stated in terms so abstract as I have just used. To give the statement life and colour, let me anticipate what will be explained in much more detail later, namely, that the most essential part of a living cell – the chromosome fibre may suitably be called an ***aperiodic*** crystal. In physics, we have dealt hitherto only with periodic crystals (Schrodinger, 1951).

Interestingly, this hypothesis has been confirmed since Watson and Crick (1953) observed the structure of the DNA.

Furthermore, later on, it was shown that quasicrystals occur in nature. They have a much more irregular structure as opposed to just simple crystals with regular structure and can be described using Penrose's hierarchical pentagonal structure that he used to tile a plane non-periodically (without global symmetry that repeats the same patterns again and again with a regular period), see Fig. 4.

Furthermore, many structures in nature exhibit fractal behavior that is impossible to describe with the classical tools of the mathematical theory of symmetry. For instance, see Fig. 5. Fractals are objects characterized by self-similar patterns that repeat at different scales. The concept of fractal, first introduced by Mandelbrot (1982), has revolutionized our understanding of natural patterns and structures. Since then, fractals have been found in various natural phenomena, from coastlines and mountains to plants and biological systems. Examples include the branching patterns of river systems, the distribution of earthquakes, the formation of coastlines, and the jagged profiles of mountain ranges, branching patterns of blood vessels, the organization of neural networks, and the growth patterns of plants. These examples are described in 3Blue1Brown (2017), Bassett and Bullmore (2006), Bullmore and Sporns (2009), Falconer (1985), and Mandelbrot (1982).

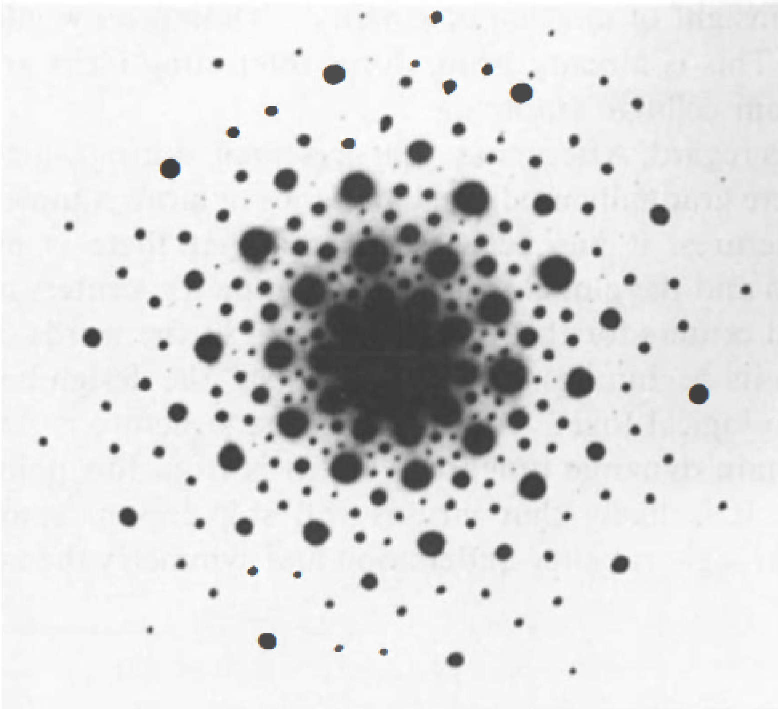

Figure 4: Diffraction pattern of quasicrystal that has both global symmetries and local symmetries (see various sizes of pentagons all around). After Senechal (1989).

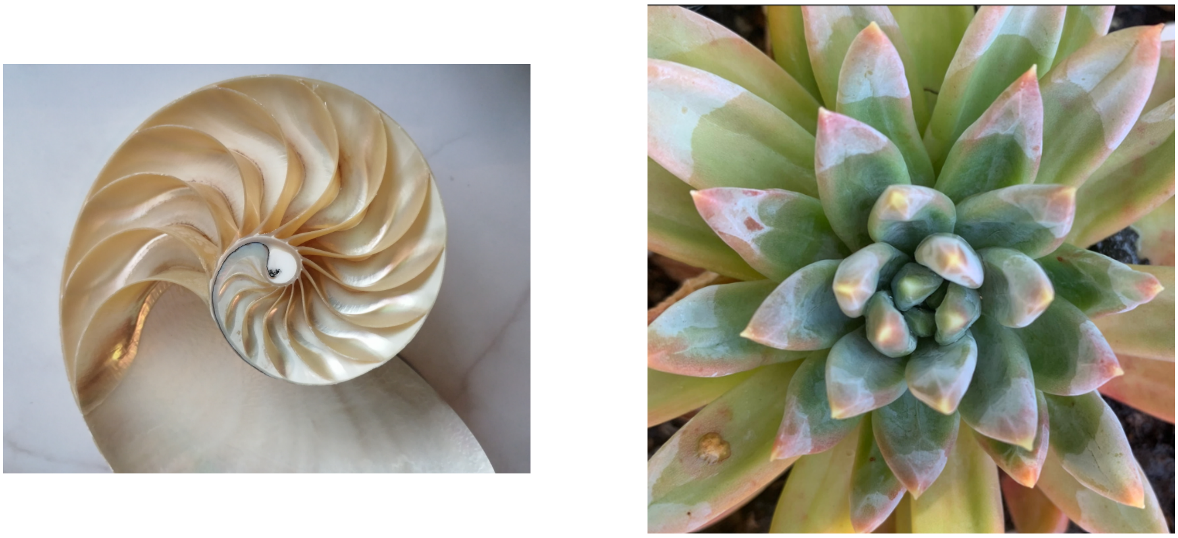

Figure 5: Examples of structures (a shell and a plant) that have fractal proportionate structure as opposed to global symmetry.

Chapter 1

Art & Morality

Symmetry, although mathematically fascinating, also has a coldness, a rigidity, a fixity, a sense of stasis, which is less interesting, less attractive, indeed less beautiful than asymmetry. Too much asymmetry is however mere chaos.

McManus

Although symmetry is an appealing concept, there is evidence that asymmetry exists not only in the subatomic world of physics (see Section 0.2) and evidently throughout the biological world, but is also utilized and developed in the arts and even in religiosity and morality.

In this chapter, we argue that from the perspective of arts and aesthetics, symmetry is cold, rigid, and static, whereas the concept of partial symmetry in the sense of proportion and local symmetries, brings life and an acceptable rate of chaos. This idea has been known to artists for a long time, see review in al-Rifaie et al. (2017) and Brandmüller and Claus (1982).

1.1 Art's structure

In this section, we explore how the concept of (partial) symmetry and asymmetry arise in the structure of art pieces.

Artists and writers may intentionally use symmetry breaking, and break or disrupt the symmetry to highlight contrasts, provoke thought, and emphasize the complexity and diversity of human experience (N. Bebiano, 2022). For instance, during my high school, I read parts of William Blake's *Songs of Innocence and of Experience* and O. Henry's short story *The Gift of the Magi*, which are prime examples of how symmetry mirroring can be employed in literature to evoke powerful effects.

O. Henry's short story The Gift of the Magi, published in 1905, centers on a young couple, Jim and Della, who each make a significant personal sacrifice to buy a Christmas gift for the other. The narrative structure of the story is partially symmetrical, as both characters' storylines follow a similar structure, selling something of worth to themselves to buy a gift for the other. This culminates in the revelation of their respective sacrifices. However, at last, it is revealed that their sacrifices have inadvertently rendered each other's gifts useless – Jim sells his watch to buy combs for Della's hair, while Della sells her hair to buy a chain for Jim's watch. This irony of the situation emphasizes the selflessness and love that motivated their actions. It is this irony that invites us to consider the true value of sacrifice and the deeper meaning of gift-giving, transcending the material objects exchanged. We might say that the valuable part of symmetry was not in the exchange of material objects but in their deeds of love.

In William Blake's Songs of Innocence and of Experience, published in 1794, Blake presents a series of paired poems that juxtapose innocence and experience, exploring the dual nature of human existence. One of the most famous poems in this collection is *The Tiger*, which is often read in conjunction with its counterpart, *The Lamb*. Each poem consists of a series of rhetorical questions about the nature of their respective subjects, i.e., the Tiger and the Lamb. The Lamb represents innocence, gentleness, and divine creation, while the Tiger explores the darker aspects of experience, power, and the potential for destruction. These dual sides of existence seem to indicate a sort of balance and complexity of the world. It is this complete inverse symmetry between dual sides, that creates a nice sense of balance and relation to the whole. Each of the poems separately also shows another obvious inner symmetry once we look at the rhymes and repetitions, which show a kind of translational approximate symmetry and unison (N. Bebiano, 2022).

Note that, in both of these stories, symmetry is just a mean towards the end. The authors used symmetry to arrive at the unification of the whole. Now, imagine having just the Tigers or just the Lambs, the authors would be unable to make contrasts, and it would tend to be boring. It is this inverse contrast that creates change and meaning. Nevertheless, even this inverse tends to be repetitive, and after one reads several stories and sees some pieces of art that have this feature, it seems boring. See for instance Fig. 1.1, while initially, these simple patterns might be nice, they are simple and static. Weyl (1989) provides many similar examples.

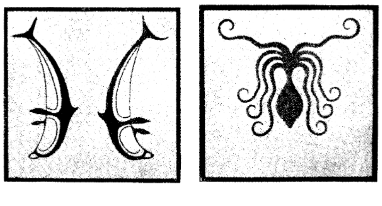

Figure 1.1: Crossing the Aegean, we find these floor patterns at the Megaron in Tiryns, late Helladic about 1200 B.C. After Weyl (1989).

With the addition of ***proportionality***, the art becomes more complex and living. Interestingly, western Muslim art already utilized a version of Penrose tilings in the Middle Ages (Makovicky, 2021). We describe these Penrose tilings in Section 3.4.2. These tilings are known as *Girith tiles* and are found in various mosques in Portugal, Spain, and Morocco, see for instance Fig. 1.2. These tilings usually incorporate both hierarchical structures of local symmetries and global symmetries. Obviously, fractals are another step that shows that the art that incorporates self-similarity and proportionality is much more living, and aesthetically enjoyable.

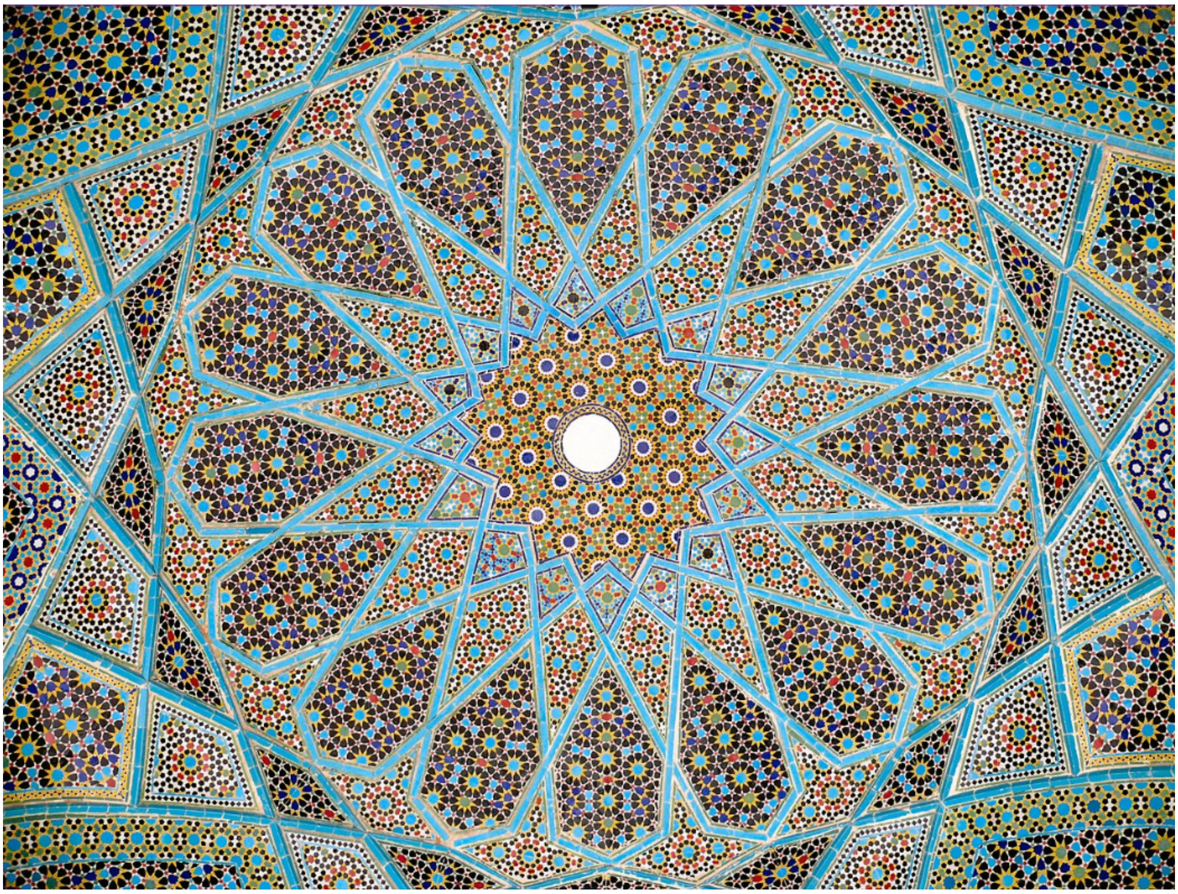

Figure 1.2: Western Muslim art utilized Penrose tilings in the Middle Ages. Source: Wikipedia contributors. (2008). *Girith Tiles.* In *Wikipedia*, Retrieved 5/19/2023, from https://upload.wikimedia.org/wikipedia/commons/thumb/6/60/Roof_hafez_tomb.jpg/1912px-Roof_hafez_tomb.jpg.

In music, Sylvestre and Costa (2010) argued that musical genius Johannes Sebastian Bach might have intentionally utilized the idea of Golden ratio, which is the definition of perfect self-similarity and proportionality, to create his *The Art of Fugue* in 1740s. They conducted an analysis of The Art of Fugue, revealing an architecture based on the golden ratios among the combinations of counterpoints and canons, see Fig. 1.3 and Fig. 1.4. The researchers examined the number of bars[1] in each piece at three known

[1] A bar is a segment of time in a piece of music that encompasses a specific number of beats. In

levels, noting a fractal property of self-similarity at varying degrees of detail due of Golden ratio proportions.

number	*BWV classification*	*Name*	*type*	*total bars*
1	1080/1	*Contrapunctus 1* [I]	simple fugue	78
2	1080/2	*Contrapunctus 2* [III]	simple fugue	84
3	1080/3	*Contrapunctus 3* [II]	simple fugue	72
4	1080/4	*Contrapunctus 4*	simple fugue	138
5	1080/5	*Contrapunctus 5* [IV]	counter-fugue	90
6	1080/6	*Contrapunctus 6 a 4 in stylo francese* [VII]	counter-fugue	79
7	1080/7	*Contrapunctus 7 a 4 per augment. et diminut.* [VIII]	counter-fugue	61
8	1080/8	*Contrapunctus 8 a 3* [X]	triple fugue	188
9	1080/9	*Contrapunctus 9 a 4 alla duodecima* [V]	double fugue	130
	1080/10*a*	*Contrapunctus a 4* [VI]	double fugue	100
10	1080/10	*Contrapunctus 10 a 4 alla decima*	double fugue	120
11	1080/11	*Contrapunctus 11 a 4* [XI]	triple fugue	184
12	1080/12,1 and 2	*Contrapunctus inversus 12 a 4* [XIII] and *Contrapunctus inversus a 4* [XIII]	mirror fugue	56 (*rectus*) 56 (*inversus*)
13	1080/13,1 and 2	*Contrapunctus a 3* [XIV] and *Contrapunctus inversus a 3* [XIV]	mirror and counter-fugue	71 (*rectus*) 71 (*inversus*)
–	1080/14	*Canon per augmentationem in contrario motu* [XII; XV; App. 1]	canon	109
–	1080/15	*Canon alla ottava* [IX]	canon	103
–	1080/16	*Canon alla decima contrapunto alla terza*	canon	82
–	1080/17	*Canon alla duodecima in contrapunto alla quinta*	canon	78
–	1080/18,1 and 2	*Fuga a 2 clav.* [App. 2] (*rectus*), and *Alio modo Fuga a 2 clav.* (*inversus*)	fugue for two instruments	71 (*rectus*) 71 (*inversus*)
14	1080/19	*Fuga a 3 soggetti* [App. 3]	(unfinished) triple (possibly quadruple) fugue	239 (ms) 233 (first ed.)

Figure 1.3: Details from the analysis of Bach's *The Art of Fugue*. After Sylvestre and Costa (2010).

Johannes Kepler's book *Harmonious world*, *Harmonices mundi* (1619) provides potential evidence that Bach might have been well familiar with the Fibonacci series and the golden ratio since Kepler's book discusses harmonic proportions in music, cosmology, and planetary motion, with multiple references to the ***divine proportion*** of golden ratio and Kepler was already well known at Bach's time.

Sylvestre and Costa (2010) argue that the precision of the ratios and the role of

musical notation, bars are separated by vertical lines called bar lines that you can usually see in musical notation.

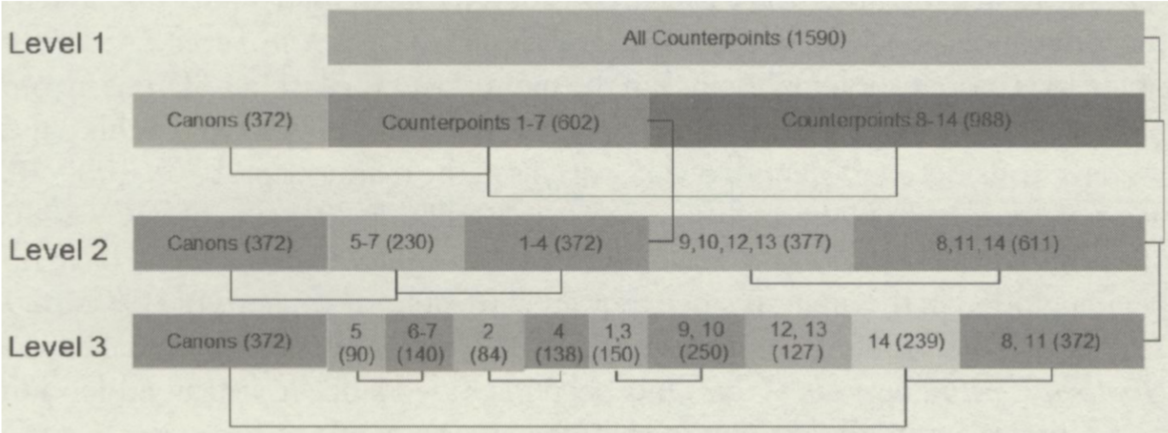

Figure 1.4: The analysis of the structure of Bach's *The Art of Fugue*. The edges link the parts of the whole whose ratio is close to the Golden ratio. After Sylvestre and Costa (2010).

Fibonacci numbers in The Art of Fugue suggest that Bach intentionally incorporated these mathematical relationships as enigmas or puzzles to convey a hidden meaning, maybe reflecting Pythagorean philosophy[2]. Obviously, the exact numeric proportions found in the work are not perceptible to listeners. Thus, if the proportions were intentional, it might likely indicate an ideological or philosophical rationale behind its mathematical architecture.

1.2 Partial symmetry and beauty

From the perspective of art, perfect lawfulness (symmetry) is "the death of man", ultimate boredom.

Fyodor Dostoevsky explores, in his work *Notes from the Underground*, the inherent limitations of rationalism and the risks associated with reducing human behavior to a set of predictable, deterministic laws through the lens of the protagonist, a man living in the underground. This work, published in 1864, has been seen as an early example of existentialist thought, critiquing the idea that human beings can be governed solely by reason and logic (Dostojevskij, 1998).

Dostoevsky's central argument is that complete perfect laws (symmetries in the modern sense), or a fully deterministic understanding of human behavior, would ultimately rob individuals of their freedom and humanity. He suggests that if human actions could be accurately predicted and governed by a strict set of rational principles, people would lose their capacity for free will and self-determination. This loss, in turn, would lead to the "death of man" in a metaphorical sense, as individuals would be reduced to mere automatons, stripped of their ability to make meaningful choices or

[2]Pythagoreans used to see a secret meaning in the numbers and proportions (Vernant, 1984).

express their unique individuality.

The Underground Man's disdain for this deterministic vision of society is also evident in his exploration of the metaphor of "the Crystal Palace," a place for the idealized, completely rational (deterministic) society envisioned by some 19th-century utopian thinkers. To him, the Crystal Palace represents a prison, as it would ultimately limit human freedom and creativity, and would be incapable of accommodating the diverse and unpredictable nature of human desires and aspirations.

We argue that it is this extreme lawfulness that our strife for perfect symmetry actually brings. Modern science is built on this search for fully deterministic laws, and thus perfect symmetries. Nevertheless, the research clearly shows that such a search might be futile. Modern quantum physics seems to indicate a kind of intrinsic random choice of nature. Recall that Einstein persistently disagreed with this intrinsic randomness of nature, and he famously said that "God does not play dice" (Zeilinger, 2005). Furthermore, we have accumulated various kinds of indications that nature tends to break these perfect symmetries, and 'utilizes' phase transitions, chaos, and fractality to produce complexity.

As opposed to classical symmetry, we rather suggest that the detection and appreciation of partial symmetries may contribute to the aesthetic experience. Penrose (1974) discussed the idea that partial symmetries bring unexpectedness, and yet local order that causes aesthetic experience. He examines various patterns and their aesthetic appeal (structurally similar to patterns in Fig. 4.1 in Chapter 4 and his tilings), noting that the ***unexpected simplicity*** might contribute to the aesthetic value of a pattern. He thinks that the idea of beauty is not in perfect symmetry which is boring since it is fully predictable, but rather the idea of beauty is in unexpected simplicity, where something that appears complicated turns out to be much simpler than anticipated, and it is this realization that leads to a sense of pleasure.

While I agree with Penrose that the emergence of unexpected simplicity can indeed bring about a sense of pleasure in some cases, I would argue that beauty is more intrinsically connected to ***dynamic interplay between unpredictability/chaos and predictability/order***, rather than solely relying on the element of unforeseen simplicity. For instance, I am thinking of any simple math problem that I struggled to solve when I was young, and I suddenly realized how simple it was, does not mean that the problem was beautiful. Now, think of any interesting fractal objects, which are commonly referred to as beautiful. What makes them beautiful is the dynamic interplay between order and chaos on the border, where one wants to zoom in and look for ever-changing, yet familiar patterns. Recall, that something similar was claimed by Montesquieu, who considered beauty a balance between order and chaos.

For instance, let me appeal to the beauty of human faces, see Fig. 1.5. It is commonly said that the more symmetrical faces are more attractive, but is it that way? I argue that the picture of the face of a young lady in the middle is much more attractive since it is much more natural and interesting than either the last or the first picture. The generated picture on the left is just too much asymmetric, and therefore strange in the negative sense. The face on the right is perhaps nice, but it is just too much bilaterally symmetric (the left side of the face is the perfect copy of the right side), and therefore artificial. As opposed to the face in the middle, the right one does not have some kind of unique symmetry-breaking point and therefore does not provide any kind of focal point for attention. The face in the middle has a birthmark and slight asymmetry in the hair, which brings a sort of life, uniqueness, and beauty as opposed to the right one. On the other hand, the left one has just too many of these asymmetries. Generalizing, we would expect existence of both lower and upper thresholds on subjective evaluation of beauty in terms of symmetry and asymmetry.

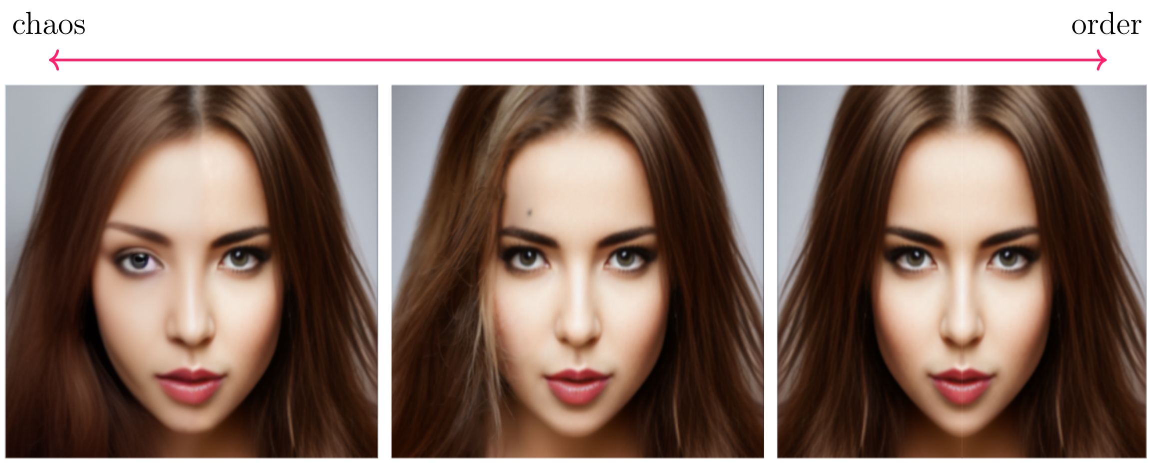

Figure 1.5: The faces of a young woman ordered in the sequence from the asymmetric face on the left to perfectly symmetric on the right. All of these images were generated with the help of Stable Diffusion, an AI technique.

1.3 Symmetry, transcendence, and morality

In this section, we predominantly follow reviews of McManus (2005) and Weyl (1989), who analyzed the completely inverse symmetric symbolic meaning of left and right.

While the origins of symbolic meanings associated with left and right are not entirely understood, they likely possess a universal human aspect. We already find an indication of this universality in the *Metaphysics*, where Aristotle discusses the Pythagoreans' identification of ten first principles (Aristotle, 2016), which include right and left, see Fig. 1.6. This suggests that this distinction was already well-known at that time.

Limited	Unlimited
Odd	Even
One	Plurality
Right	**Left**
Male	Female
At rest	In motion
Straight	Crooked
Light	Darkness
Good	Evil
Square	Oblong

Figure 1.6: The ten Pythagorean principles of opposites described by Aristotle according to McManus (2005)

Moreover, the Purum, a tribe residing on the Indo-Burmese border, also exhibit a variety of left-right symbolism connected to transcendence and morality (McManus, 2005), see their dualistic symbolic representation in Fig. 1.7.

Right	Left	Right	Left
Male	Female	Kin	Affines
Masculine	Feminine	Private	Public
Moon	Sun	Superior	Inferior
Sky	Earth	Above	Below
East	West	Auspicious	Inauspicious
Life	Death	South	(North)
Good death	Bad death	Sacred	Profane
Odd	Even	Sexual Abstinence	Sexual activity
Family	Strangers	Village	Forest
Wife givers	Wife takers	Prosperity	Famine
Gods, Ancestral spirits	Mortals	Beneficent spirits	Evil spirits, ghosts
Back	Front		

Figure 1.7: “The dual symbolic classification of the Purums in relation to right and left” (McManus, 2005)

Furthermore, McManus (2005) shows consensus about the phenomenological properties of the concepts of symmetry and asymmetry (right and left) according to art historians and philosophers where symmetry is associated with order, but boredom, and asymmetry with life, freedom, complexity, surprise, and chaos, see Fig. 1.8.

Interestingly, this consensus well agrees with our modern understanding of morality, religiosity, and mythology. Eliade (2022) and Peterson (2002) argue that ancient myths, moralities, and religions roughly follow this distinction between left and right, chaos and order. However, myths even provide a universal way how to lead a life by uniting these opposite forces. Recall the middle way of Buddhists, Chinese Ying and Yang, virtuous mean of ancient Greeks such as in Aristotle and Christianity, etc.

Symmetry	Asymmetry
Rest	Motion
Binding	Loosening
Order	Arbitrariness
Law	Accident
Formal rigidity	Life, play
Constraint	Freedom
Boredom	Interest
Stillness	Chaos
Monotony	Surprise
Fixity	Detachment
Stasis	Flux
Simplicity	Complexity

Figure 1.8: This image summarizes the aesthetic and psychological properties associated with symmetry and asymmetry according to art historians and philosophers. After McManus (2005)

A similar meaning of balance between left and right is even incorporated in the medieval architectural and symbolic design of churches (see Fig. 1.9) where church was seen as a meeting of eternal and mortal, and art (Michelangelo's *The creation of Adam*, and see also Fig. 1.10).

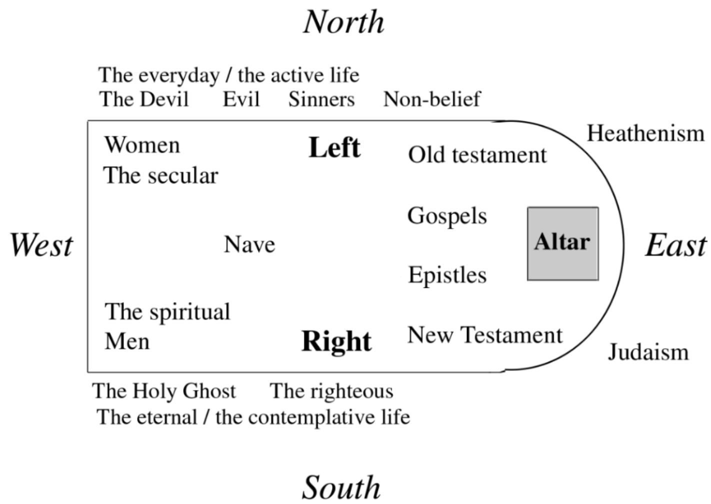

Figure 1.9: This picture shows a symbolic understanding of the orientation of churches in Christian architectural symbolicism. After McManus (2005).

This universality of the tendency towards inverse symmetrical balance between left and right might be driven by brain/body symmetries (McManus, 2005; Peterson, 2002; Wagemans, 1997), see Fig. 1.11. This hypothesis makes sense, since even our brain is inverse symmetrical in terms of both structure and function (Gazzaniga, 2000; Peterson, 2002). We explore this tendency from a cognitive perspective in the next chapter.

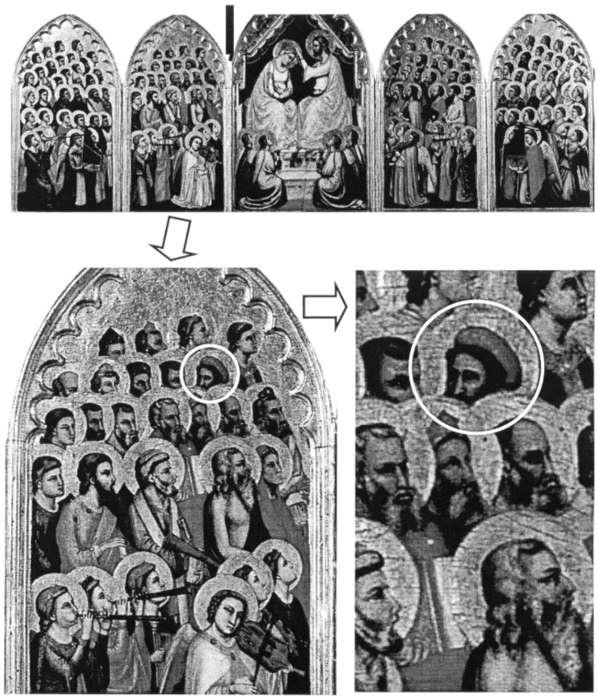

Figure 1.10: This picture, *Broncelli Polyptych* by Giotto and his school (1334), shows a tendency of arts to utilize the symmetry breaking and mirror images between left and right to signal a religious message in Christian medieval art. After McManus (2005).

Figure 1.11: This picture shows a common way of looking at the asymmetry of brain structure and function among cognitive scientists and psychologists. While holistically, the structure seems highly symmetrical, the functions tend to be highly asymmetrical, inverse. After Peterson (2002).

Many examples in this chapter suggest that the concept of perfect bilateral symmetry and mirror-image balance might indeed rather be an artifact of our minds that we use in art and our mythological descriptions of reality rather than the intrinsic feature of nature.

Chapter 2

Cognition

What opposes unites, and that the finest attunement stems from things bearing in opposite directions.

Heracleitus

The concept of symmetry is related to cognition in several ways, as we will show in the following sections. Moreover, we will further try to demonstrate that perhaps the concept of perfect symmetry is an idealization and construct of our minds than the essential feature of nature, as was indicated in the previous chapter.

2.1 Symmetry detection

Bornstein et al. (1981) investigated infants' perception of symmetry, finding that "younger infants habituate fastest and most to vertical symmetry" but do not show a preference for it until they are older. Measurements were done on 4-months old infants and 12-months old infants. For infants as old as 4 months, the preference for detecting symmetry in terms of mean looking time was not detected. However, it was detected for the 12-months old infants, see Fig. 2.2. Interestingly, infants habituated faster to bilateral symmetries already when they were 4-months, but only to bilateral symmetry, not to horizontal symmetry, see Fig. 2.1. This is possibly due to the bilateral symmetry of the visual system, as speculated by Mach (1959), or their innate tendency to treat symmetric patterns about the vertical axis as equivalent. The later development of a preference for vertical symmetry may stem from maturation, exposure to a symmetry-dominated visual environment, or an appreciation of the benefits of information redundancy provided by symmetry earlier in life. In either of these explanations, these experiments support the idea that bilateral symmetry has a special status in perceptual development (Bornstein et al., 1981).

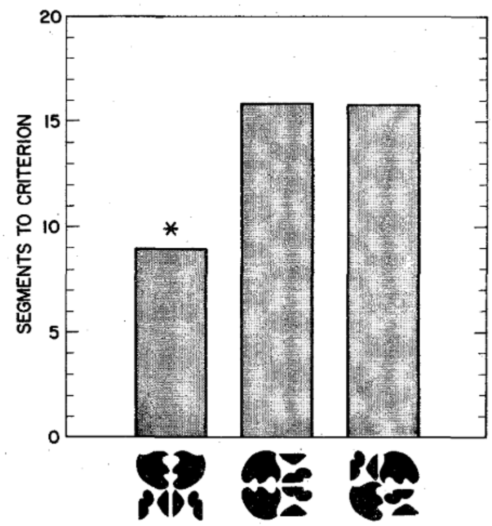

Figure 2.1: "Mean number of 10-sec segments prior to reaching criterion for habituation to the vertically symmetrical, horizontally symmetrical, and asymmetrical stimuli. (The asterisk indicates that the 4-month-olds habituated significantly faster to vertical symmetry than to horizontal symmetry and asymmetry)" (Bornstein et al., 1981).

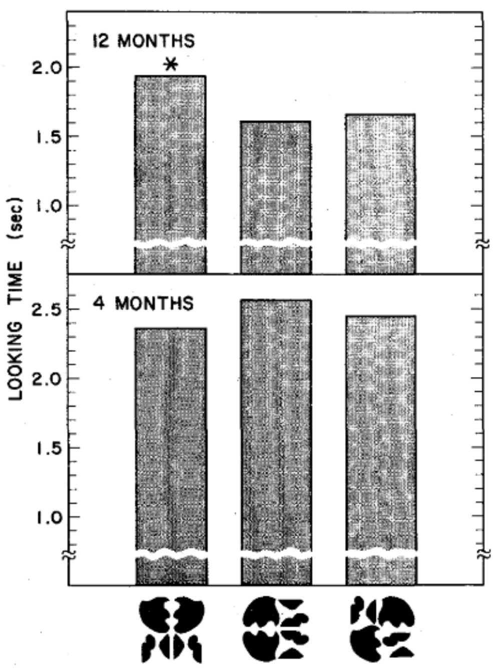

Figure 2.2: "Mean looking time of 4-month-old and 12 month-old infants at vertically symmetrical, horizontally symmetrical, and asymmetrical stimuli. (The asterisk indicates that the 12-month-olds significantly preferred vertical symmetry to horizontal symmetry and asymmetry" (Bornstein et al., 1981).

Further, Wagemans (1997) reviewed the state of knowledge of symmetry detection. He confirms that people have a preference for mirror symmetry (reflection) over other types of symmetries like translation or rotation, as mirror symmetry detection is relatively effortless, supporting findings in infants that we discussed above. Further, he highlights that mirror symmetry can be detected preattentively, meaning it does not always require focused attention.

Moreover, Wagemans (1997) discusses the possibility of a connection between the preference for mirror symmetry and the mirror symmetry of the brain itself. However, this hypothesis has not yet been confirmed, and it is probably false since from the neuroscientific point of view symmetry detection tends to be lateralized to the right brain lobe (Bertamini et al., 2018; Cattaneo, 2017).

Furthermore, Wagemans (1997) suggests that various factors, such as the orientation of the symmetry axis, proximity to the axis, and pattern density, impact the detection process. Nevertheless, symmetry detection is found to be both robust, allowing for the categorization of less than perfectly symmetrical patterns as symmetrical, and sensitive, capable of discerning minute perturbations from perfect symmetry if asked for.

From the neuroscientific point of view, our ability to perceive symmetries is facilitated by a combination of low-level visual processes and higher-order cognitive processes such as the right occipital face area (OFA) and lateral occipital (LO) complex (Bertamini et al., 2018; Cattaneo, 2017). Early visual processing stages detects basic symmetrical features in the environment, while higher-order cognitive processes integrate this information and generate mental representations of the symmetrical structures.

Interestingly, possibly distinct cortical networks are involved in processing different types of visual symmetry. Cattaneo et al. (2017) found that the ability to detect vertical symmetry was negatively affected by TMS applied to both LO and OFA brain regions, while the detection of horizontal symmetry was only delayed when LO was stimulated.

From the point of view of machine learning, as of 2017, symmetry detection is not yet a solved problem. Ke et al. (2017) highlight that symmetry detection can improve various computer vision tasks, such as image segmentation, or foreground extraction. They introduce a new benchmark with complex backgrounds and an end-to-end deep symmetry detection approach for processing photos. In Fig. 2.3, we can see that their SRN outperforms other models. However, as the authors note, this problem of symmetry detection of real images is far from being solved since even the best models achieve maximally around $0.6 - 0.7$ F-measure[1] on most of the datasets (Ke et al., 2017).

[1]The F-measure is used to evaluate the performance of neural networks in classification tasks. Higher F-measure score indicates better performance of the neural network. Using harmonic mean, it combines precision and recall into a single score, which ranges from 0 to 1. Precision is the ratio of true positive predictions to the total number of positive predictions, while recall is the ratio of true positive predictions to the total number of positive predictions in the dataset.

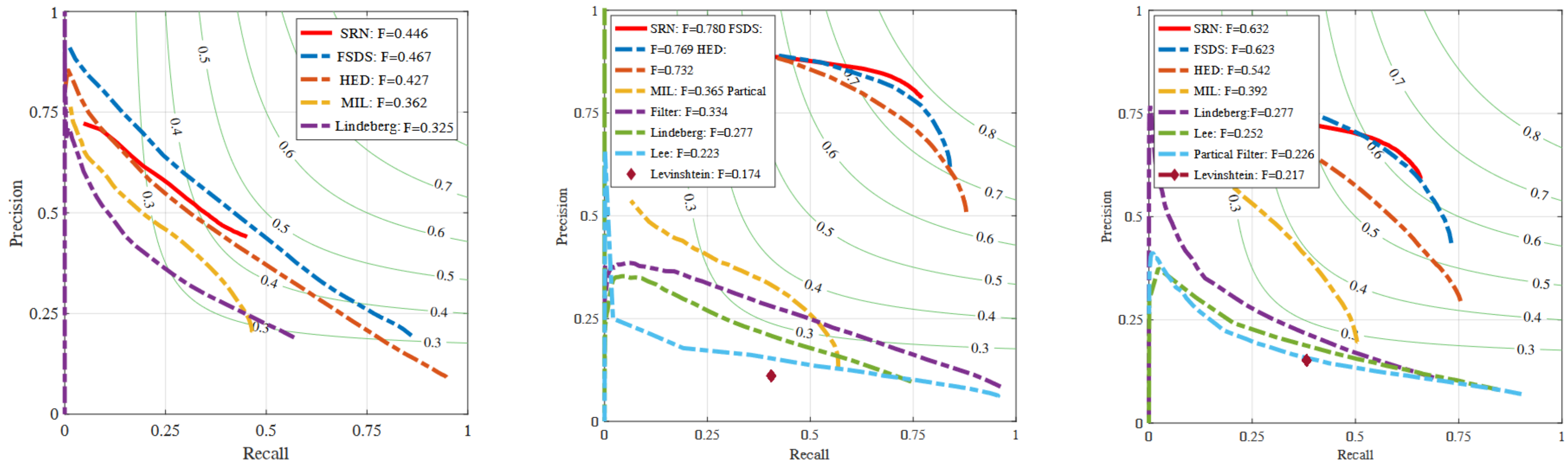

Figure 2.3: Performance of different models on symmetry detection of real images. After Ke et al. (2017)

2.2 Perceptual organization

Symmetry is a fundamental concept in the theory of perception. Perceptual constancy refers to the fact that the perceived geometrical and physical characteristics of objects remain constant despite transformations of the objects, such as rigid motion (Friedenberg & Silverman, 2006). Formally, perceptual constancy and permanence of objects are invariants.

However, this perceptual constancy relates not only to the one that we would describe using formalization of the modern concept of symmetry in group theory[2]. Consider that objects are perceived as invariant if you gradually increase the distance between the observer and the objects, and the object is proportional to the original one. This is the constancy of the proportions of the objects. However, while the perspective of group theory would assign the same global symmetry to the object, it would not be able to completely capture this process of change, and proportion between the original object, and the one moved into the distance.

Moreover, the Gestalt principle of symmetry is one of the core Gestalt principles that describe how humans tend to organize visual information. According to this principle, elements that are symmetrical to each other tend to be perceived as a unified whole (Koffka, 1936; Todorovic, 2008; Wertheimer, 1938). This means that when we see objects that are symmetrical with each other, we are more likely to group them in our minds than objects that are not symmetrical with each other. The principle is related to our tendency to look for order and balance in visual compositions. Symmetry is one way of achieving this balance.

Recall that Wagemans (1997) reviewed the symmetry detection research and noted that the symmetry detection is however robust to slight asymmetries. This suggests that the Gestalt principle of symmetry is robust as well. Moreover, this indicates that the principle is perhaps better described using approximate partial symmetries than perfect global symmetries.

Furthermore, I think that there might be a possibility of the existence of the 'principle of Gestalt fractality'. See, for instance, Fig. 3.15 in Section 3.4.2, if the dots are well proportioned, the figure seems to organize as a whole depending partly on global features, but on proportion as well. The proposed principle of Gestalt fractality could be conceived as a phenomenon where the Gestalt principles apply at various levels of perceptual processing, creating a kind of 'nested' perception. This implies that a complex perceptual whole can be perceived similarly at different levels of resolution or scale, akin to a fractal's self-similarity across scales. Such an idea would extend the applicability of Gestalt principles beyond immediate perceptual wholes, suggesting that

[2]We suggest the reader return to this point later, once the formal aspects of group theory and inverse semigroups are described.

these principles are deeply ingrained in our perceptual systems, operating consistently at various levels of perceptual processing. Moreover, Friedenberg et al. (2022) in their experimental study suggest that people may have a preference for complex fractal patterning, likely due to evolution in a natural world full of such patterns. They also introduce the idea that this preference could be due to the efficient neural computation these patterns allow, or the pleasurable sensation they may create through neural synchrony. However, they do not yet call it a new Gestalt principle. We are not sure whether such a new Gestalt principle would be useful. It might be reducible to other Gestalt principles, but it might be worth looking into.

Finally, neuroaesthetics provides a bridge between psychology, neuroscience, and art that we already crossed. Neuroaesthetics is an emerging field within cognitive neuroscience focused on understanding the biological basis of aesthetic experiences, which involve appraisals of natural objects, artifacts, and environments (Chatterjee & Vartanian, 2014).

In their study, Ishizu and Zeki (2011) hypothesized that a single brain area or set of areas would correlate with the experience of beauty, regardless of the source. They explored the neurobiological basis of beauty perception through functional magnetic resonance imaging (fMRI) and conducted experiments with 21 healthy right-handed volunteers, who were asked to rate the beauty of visual and auditory stimuli.

Their findings revealed that the experience of beauty correlates with activity in the medial orbitofrontal cortex (mOFC), specifically in field A1. Thus, the authors propose a neurobiological definition of beauty that shifts the focus from the perceiving object. Such a definition has the benefit of being indifferent to what is considered art or not art.

The authors acknowledge that activity in the mOFC has been associated with pleasure and reward experiences in other domains. However, they argue that the co-activation of field A1 in mOFC with specialized sensory and perceptive areas is the determinant of beauty. This broader definition includes both the activation of mOFC and its co-activation with sensory areas that feed it.

Interestingly, these identified brain areas associated with beauty do not match the areas associated with symmetry detection that were reviewed in the previous section, except perhaps for the low-level visual processing. This suggests we should question the common narrative saying that symmetry is a marker of beauty.

2.3 Symmetry and thought

Our affinity for perfect symmetry has been observed in many aspects of human life, including science, art, and architecture. Perhaps, this tendency to look preferentially for perfect symmetry and balance in the world around us is a fundamental aspect of human cognition, not of the world itself. This tendency is reflected in many areas of human thought and behavior, including our perception of visual information, understanding of the physical world, and even our social interactions. Recall, for instance, the concept of the "Law of Good Gestalt" in Gestalt psychology stating that humans have a natural tendency to organize complex visual stimuli in the simplest, most coherent, and symmetrical manner possible (Koffka, 1936). Our cognitive processes aim to generate a good gestalt or a well-organized, harmonious, and balanced perception of the world. Mach (1959) had a similar idea when it comes to thought. He proposed a principle of "economy of thought," a principle that suggests that our cognitive processes, including perception, thinking, and problem-solving, naturally seek the most efficient and straightforward ways to process and organize information similar to the medieval principle of "Ockham's razor". Symmetries may play a crucial role in the mental representations that our minds create to understand and navigate the world. By simplifying complex patterns and structures, symmetries enable us to process and categorize information more efficiently.

Even though, our preference for symmetry may have evolved as a result of various adaptive advantages. It is commonly stated that symmetry is an indicator of health and genetic fitness in potential mates, as well as a sign of stability and strength in the natural environment. This evolutionary pressure may have led to the development of cognitive mechanisms that specifically process and appreciate symmetry.

Alternatively, it might have simply spontaneously emerged from the algorithmic nature of evolution in the sense of efficient information encoding. Johnston et al. (2022) investigate this alternative hypothesis and they found that

> symmetric structures preferentially arise not just due to natural selection but also because they require less specific information to encode and are therefore much more likely to appear as phenotypic variation through random mutations. Arguments from algorithmic information theory can formalize this intuition, leading to the prediction that many genotype–phenotype maps are exponentially biased toward phenotypes with low descriptional complexity. A preference for symmetry is a special case of this bias toward compressible descriptions. We test these predictions with extensive biological data, showing that protein complexes, RNA secondary structures, and a model gene regulatory network all exhibit the expected exponential bias toward simpler (and more symmetric) phenotypes. Lower descriptional complexity also correlates with higher mutational robustness, which may aid the evolution of complex modular assemblies of multiple components.

This is an interesting hypothesis which they robustly back up by analysing huge parts of various phenotype databases. I find this as another indication that there is a tendency in living organisms for perfect symmetries.

Despite this preferential evolving towards symmetrical shapes, our brains not only have approximate bilateral symmetry, but also have scale-free/fractal-like proportionate structure, i.e., neural networks have big well-interconnected hubs, and there is a short average distance between nodes to transmit information (Bassett & Bullmore, 2006). Bullmore and Sporns (2009) review the exploration of structural and functional brain networks using graph theory, which involves four main steps: 1) defining network nodes (for instance, individual electrodes/pixels might correspond to nodes), 2) estimating continuous measures of association between nodes, 3) generating an association matrix and applying a threshold to create a binary adjacency matrix or undirected graph, and 4) calculating network parameters and comparing them to random network parameters, see all these steps illustrated in Fig. 2.4. Each step entails choices that must be carefully informed by the experimental question. Despite the difficulties, they review several studies and identify scale-free fractal-like structures of both structural and functional neural networks.

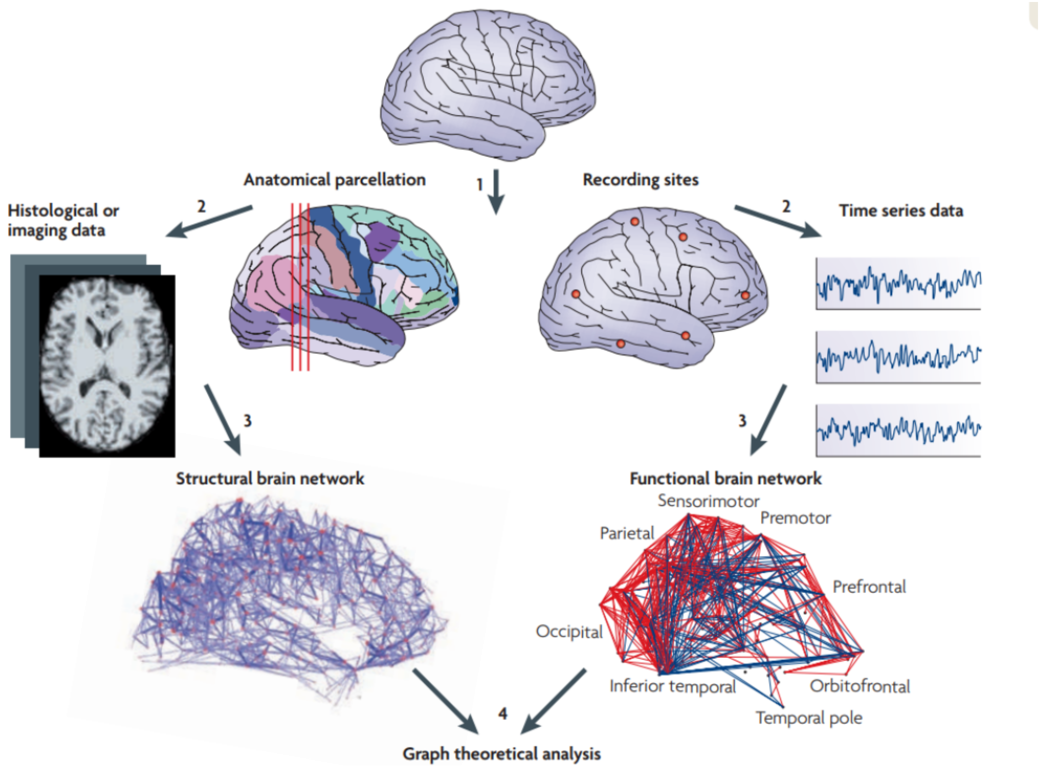

Figure 2.4: Illustration of the process of translating neuroscientific recordings into networks for further graph analysis. After Bullmore and Sporns (2009).

Bassett and Bullmore (2006) and Bullmore and Sporns (2009) suggest that spatially embedded complex (neural) networks may have evolved to optimize information transfer efficiency at low connection cost or achieve an optimal balance between functional segregation and integration for high complexity dynamics. They argue that if wiring cost was exclusively prioritized, the network would be close to a regular lattice, whereas if efficiency was the only criterion, the network would be random. A key issue for future research in this area is to understand how functional networks interact with structural substrates and how functional network topology changes over time.

These considerations about the self-similar nature of our brains suggest that proportional self-similarity, which might be capturable by the concept of partial symmetry as we argue in the next chapter, might have to do something with our brains, even though, our minds tend to idealize things towards perfect symmetry.

Chapter 3

Mathematics

The chief forms of beauty are order and symmetry [summetria] and definiteness, which the mathematical sciences demonstrate in a special degree.

Aristotle

Interestingly, Hargittai (1989) compiled one of the most extensive volumes written to synthesize knowledge on global symmetries as used in various sciences, and the compilation starts with an article criticizing the concept of the global symmetry as captured by classical group theory, and the need for a new mathematical tool and generalized concept of symmetry. Why is this the case, and what might be a suitable candidate for the substitution of the original concept?

In this section, we formalize the notion of global symmetries in Section 3.2 and partial symmetries in Section 3.3. The primary motivation and takeaway for this chapter is to contrast these two concepts in formalism, and to show why the modern meaning of global symmetry is very restrictive as opposed to partial/local symmetry.

We will demonstrate these concepts on the graphs. We choose graphs because they are widely used in all areas of science, from the study of the structure of molecules made up of atoms to the research on neuronal networks and more. Graphs are used to represent these structures. Thus, the study of symmetries and their properties in graph theory is vital for scientific progress.

Firstly, let us summarize the most important preliminaries that will re-occur in most of the proofs for the sake of completeness. Since this thesis is intended for a wider audience, we omit all the unnecessary algebraic mastery details of proofs, and we only assume that the reader is aware of high-school algebra, mostly the idea of permutation, basic algebraic manipulations and that the reader has no problem with formalizations and standard mathematical notations. It is possible to simply skip the proofs and try to grasp the general idea through the comments that we leave at the beginning and end of each section. We strongly suggest the reader with an undergraduate math course dealing with discrete math skip straight to the Definition 14 of a group.

3.1 Graph theory

In this section, we follow the classical textbook of Diestel (2017). Typically, a network of relations among items is represented by a graph. They are made up of edges and vertices. A vertex, also known as a node, symbolizes an object or a point (for example on a map). A connected pair of vertices is referred to as an edge. Edges depict the connections between the two vertices and are often shown as lines connecting points, while vertices are typically shown as points/circles.

Definition 1 (Graph)**.** A ***graph*** Γ is an ordered pair $(V(\Gamma), E(\Gamma))$ consisting of a nonempty set $V(\Gamma)$ of vertices, a set $E(\Gamma)$, disjoint from $V(\Gamma)$, of edges, i.e., a set of unordered pairs $\{u, v\}$, where $u, v \in V$. See examples in Fig. 3.1 and Fig. 3.3.

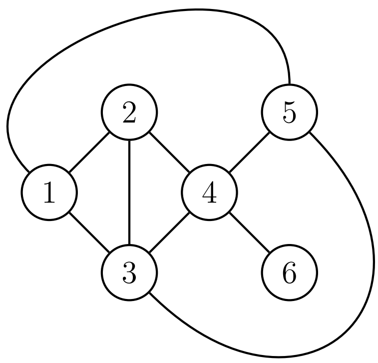

Figure 3.1: Example of graph with 6 vertices and 8 edges.

Definition 2 (Subgraph)**.** Graph $\bar{\Gamma}$ is a ***subgraph*** of graph $\Gamma(\bar{\Gamma} \subseteq \Gamma)$ when graph $\bar{\Gamma}$ consists of vertices and edges, which are in graph Γ. In other words, a vertex set of graph $\bar{\Gamma}$ is a subset of a vertex set of graph Γ and an edge set of graph $\bar{\Gamma}$ is a subset of an edge set of graph Γ such that the edges contain the only vertices from the vertex set of subgraph $\bar{\Gamma}$. See example in Fig. 3.3

Definition 3 (Induced subgraph)**.** Let $\bar{\Gamma}$ be a subgraph of graph $\Gamma(\bar{\Gamma} \subseteq \Gamma)$. If graph $\bar{\Gamma}$ is a subgraph of graph Γ and graph $\bar{\Gamma}$ also contains all edges from graph Γ connecting the vertices of graph $\bar{\Gamma}$, then graph $\bar{\Gamma}$ is called an ***induced subgraph***. In other words, induced subgraph is the same graph with the same edges if we do not consider the deleted nodes. See the difference between the examples in Fig. 3.2 and Fig. 3.3.

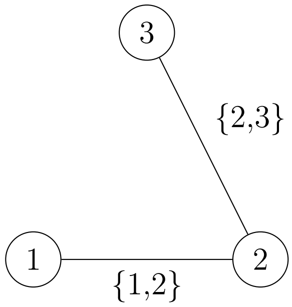

Figure 3.2: Example of subgraph of graph in Fig. 3.1 that is not induced.

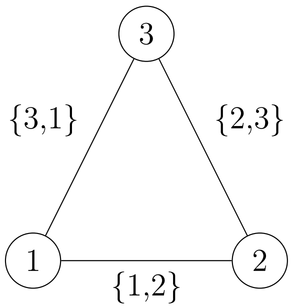

Figure 3.3: Example of a simple graph with 3 vertices that is an induced subgraph of graph in Fig. 3.1.

There are some graphs that share the same set of edges and vertices. We call them ***identical graphs***. Let $\Gamma = (V, E)$ and $\bar{\Gamma} = (V', E')$, for instance. Graphs Γ and $\bar{\Gamma}$ are the same if $V = V'$ and $E = E'$. There are other graphs, nevertheless, that share a structure but differ in their vertices and edges. We say that these graphs are ***isomorphic***.

Definition 4 (Graph isomorphism)**.** Two graphs Γ and $\bar{\Gamma}$ are ***isomorphic*** (written $\Gamma \cong \bar{\Gamma}$) if there exists a bijection[1] $\theta : V(\Gamma) \to V(\bar{\Gamma})$ such that $\{u, v\} \in E(\Gamma)$ if and only if (iff is shortcut for double-sided implication) $\{\theta(u), \theta(v)\} \in E(\bar{\Gamma})$.

In simpler terms, two graphs, Γ and $\bar{\Gamma}$, are isomorphic if there is a way to match the points (vertices) and lines (edges) of graph Γ with those of graph $\bar{\Gamma}$ while keeping their overall structure the same. Obviously, identical graphs are isomorphic. Sometimes, though, you can match the points (nodes) and lines (edges) of a graph to itself.

Definition 5 (Graph automorphism)**.** This is called ***graph automorphism***, and there is an example in the Fig. 3.4. Formally, graph automorphism means that a graph is isomorphic to itself. All graphs have also ***trivial*** automorphism, a map that maps the graph identically to itself. Any other automorphisms are ***non-trivial***. The group of all automorphisms of a graph Γ is denoted as $Aut(\Gamma)$.

We can imagine a graph as a neural network of neurons (vertices) connected by synapses and dendrites (edges). An automorphism of a graph is a way to rearrange

[1]The definition of a bijection is in the next section.

the neurons while keeping all the connections the same. In other words, the structure remains the same.

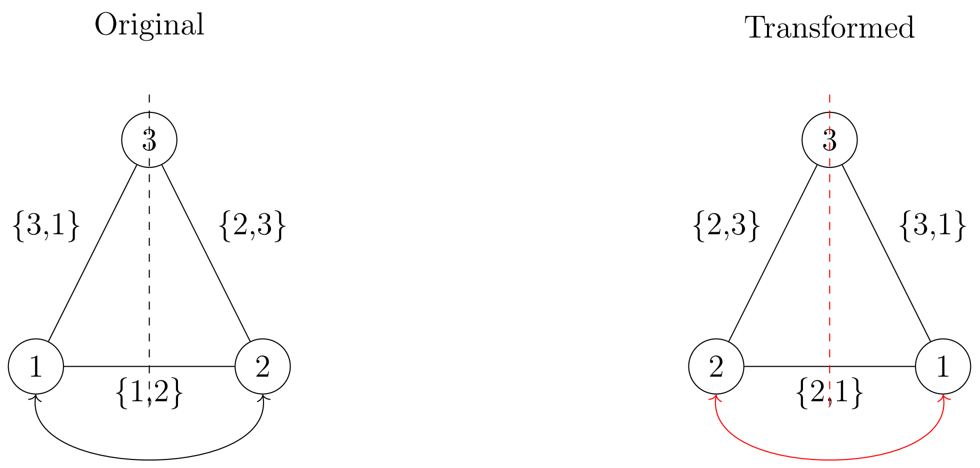

Figure 3.4: Example of a graph automorphism

Definition 6 (Partial graph automorphism)**.** A *partial automorphism of a graph* Γ is an isomorphism between two induced subgraphs of Γ. In other words, it is a bijection $\varphi : V_1 \to V_2$ between two sets of vertices $V_1, V_2 \subseteq V(\Gamma)$ such that any pair of vertices $u, v \in V_l$ satisfies the condition $\{u, v\} \in E$ iff $\{\varphi(u), \varphi(v)\} \in E$. Similarly, all partial automorphisms of any graph Γ together are denoted as $PAut(\Gamma)$.

Partial graph automorphism is similar to graph automorphism, but it does not have to involve all the vertices. As opposed to graph automorphism, it preserves only part of the whole structure. Imagine if we only rearranged (renamed) some neurons in our network, but we still kept all the connections between those neurons structurally the same, irrespective of what happens with the other neurons. This would be a partial automorphism of the graph.

Finally, let us mention the theorem of Erdős and Rényi (1963) who proved that the probability of selecting an asymmetric graph from the set of all graphs is almost 1, thus, almost all finite graphs are *asymmetric*, i.e., they have only a trivial automorphism. This further motivates our study of partial automorphisms of graphs.

3.2 Symmetry and group theory

It is a well-established fact that the modern concept of symmetry can be formalized using the theory of groups. In this section, for the sake of completeness, we provide basic preliminaries and a summary of the fundamental results of group theory following the classical textbook on abstract algebra Gallian (2021). We will provide some useful commentaries on each of these theorems and definitions to set them in context to study symmetries. The primary takeaway messages of this section are: the understanding that the concept of group covers the set of all ***global transformations*** that leave the object invariant, and that the abstract concept of a group can be represented as a permutation of the set. Let us begin with the preliminaries:

Preliminaries

Definition 7 (Mapping)**.** A ***mapping*** $f : A \rightarrow B$ is a function that assigns each element $a \in A$ to a unique element $b \in B$. Then, we call A the domain of the function f, and B the range/image of the function f.

A function is a mathematical relationship between two sets, where each input value (from the first set) corresponds to exactly one output value (from the second set). It can be represented using a graph, table, or equation.

Definition 8 (Composition of Functions)**.** Let $f : A \rightarrow B$ and $g : B \rightarrow C$. The ***composition*** $g \circ f$ or simply gf is the mapping from A to C defined by $(gf)(a) = g(f(a))$ for all a in A. The composition of function gf can be visualized as in Fig. 3.5.

Composition of functions is a mathematical operation that combines two or more functions to create a new function. It is defined as the application of one function to the result of another function. In the definition above, this means that the output of any function denoted as f is used as the input of any function g.

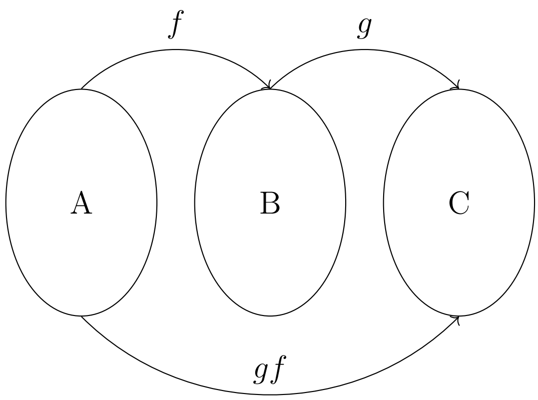

Figure 3.5: Composition of functions.

Definition 9 (Injective Function)**.** A function $f : A \rightarrow B$ is ***injective*** if for all $a_1, a_2 \in X$, $f(a_1) = f(a_2) \implies a_1 = a_2$.

An ***injective*** function is a function in which each element in the domain is paired with exactly at most one element in the range. This means that no two elements in the domain are mapped to the same element in the range. Note that not every element in the range has to have some element in the domain.

Definition 10 (Function from A onto B (surjective))**.** Let A and B be sets. A function $f : A \to B$ is a ***surjective*** function from A ***onto*** B if for every $b \in B$ there exists a $a \in A$ such that $f(a) = b$.

Surjectivity simply says that each element of the set B in the image of the function has to have at least one corresponding element mapped onto it from the domain of the function.

Definition 11 (Bijection from A to B))**.** Let A and B be sets. A function $f : A \to B$ is a ***bijective*** function iff it is both injective and surjective.

That is, each element from the domain has precisely one corresponding element in the image, and vice versa.

Definition 12 (Permutation)**.** A ***permutation*** of a set A is defined as a function that maps A to A, possessing both injective and surjective properties, thus, a bijection.

Definition 13 (Binary Operation)**.** Let A be a set. A ***binary operation*** on A is a function $f : A \times A \to A$, i.e., a function that assigns each pair of elements of A, one element of A

Examples of binary operations include addition, subtraction, multiplication, division, exponentiation, etc.

Group theory

Now, we are ready to review the results from group theory. Recall that group theory was developed as a formal abstraction of what we would conceptualize as global symmetry, a set of all transformations under which an object is invariant as discovered in the 19th century. Formally:

Definition 14 (Group)**.** A ***group*** is a set G together with a binary operation $\cdot$ that satisfies four axioms: closure, associativity, identity, and inverse. That is, for any $a, b, c \in G$, we have:

1. ***Closure***: $a \cdot b \in G$ (we usually omit $\cdot$)
2. ***Associativity***: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. ***Identity***: There exists an element $id \in G$ such that $a \cdot id = id \cdot a = a$ for all $a \in G$
4. ***Inverse***: For each $a \in G$, there exists an element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = id$

We will work with the graph in Fig. 3.3 to illustrate D_3 group, dihedral symmetry for a regular triangle. The D_3 group has six possible transformations that leave the object invariant: 0° rotation which is called identity, 120° rotation, 240°, and three different involutions that reverse the order. (geometrically, inverting image across axis going through nodes and the middle of the opposing sides, which are shown in Fig. 3.6).

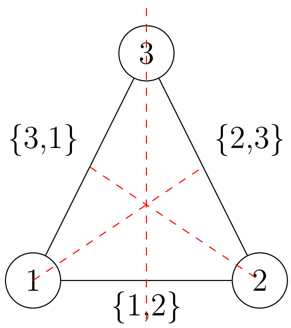

Figure 3.6: Example of a triangular graph that has the D_3 symmetry. The image highlights three possible axes for inversion of image bilaterally. 3D analogue of this graph is a tetrahedron, which is the general form of many molecules such as methane (CH_4).

These transformations can be written as permutation mappings:

- The identity permutation (no change):

$$1 \mapsto 1$$
$$2 \mapsto 2$$
$$3 \mapsto 3$$

- Rotation by 120 degrees counterclockwise:

$$1 \mapsto 2$$
$$2 \mapsto 3$$
$$3 \mapsto 1$$

- Rotation by 240 degrees counterclockwise:

$$1 \mapsto 3$$
$$2 \mapsto 1$$
$$3 \mapsto 2$$

- Reflection across the axis passing through vertex 1:

$$1 \mapsto 1$$
$$2 \mapsto 3$$
$$3 \mapsto 2$$

- Reflection across the axis passing through vertex 2:

$$1 \mapsto 3$$
$$2 \mapsto 2$$
$$3 \mapsto 1$$

- Reflection across the axis passing through vertex 3:

$$1 \mapsto 2$$
$$2 \mapsto 1$$
$$3 \mapsto 3$$

Now, consider a simple intuition for each of the axiomatic properties of a group:

1. Closure: This property means that if you take any two elements from the group and combine them using the group operation, the result will always be another element in the group. This ensures that the group is closed under its operation and that you can keep combining elements without ever leaving the group. Take for instance a regular triangle in Fig. 3.6 which has a D_3 symmetry. When you rotate this object by 240° counterclockwise and then make a mirror image of it across 2, these transformations taken together make the same transformation as if you would make an inversion across 3 instead.

2. Associativity: This property means that when you combine three or more elements using the group operation, it does not matter how you group them. This allows us to manipulate expressions involving the group operation without having to worry about the order in which we perform the operations.

3. Identity Property: This property means that there is a special element in the group called the identity element such that when you combine any element with the identity element using the group operation, you get the same element back. The identity element acts as a neutral element that does not change other elements when combined with them. It is the same as doing nothing / 0° rotation.

4. Inverse Property: This property means that for every element in the group, there is another element called its inverse such that when you combine an element with its inverse using the group operation, you get the identity element. The inverse of an element 'undoes' the effect of combining with that element. Observe this on the triangle, if you make a rotation to the right, there is a rotation to the left. If you make a mirror transformation, there is another mirror transformation that takes the first transformation back to the original one.

Definition 15 (Order of a group)**.** Let G be a group. The ***order*** of G, denoted by $|G|$, is the number of elements in G.

Now, we list some of the most essential but simple theorems. We also include proofs for the reader to see the basic algebraic manipulations in the group theory.

Theorem 1 (Uniqueness of the identity of a group)**.** *Let G be a group. Then there exists a unique identity element $id \in G$ such that for all $a \in G$, $id \cdot a = a \cdot id = a$.*

Proof. Let $id_1, id_2 \in G$ be two elements such that for all $a \in G$, we get two equations $id_1 \cdot a = a \cdot id_1 = a$ and $id_2 \cdot a = a \cdot id_2 = a$. Then, if we choose $a = id_2$ in the first equation, we get (by definition of identity in the group)

$$id_1 \cdot id_2 = id_2 \cdot id_1 = id_2$$

If we choose $a = id_1$ in the second equation, we get

$$id_2 \cdot id_1 = id_1 \cdot id_2 = id_1$$

Therefore, $id_1 = id_2$, and the element $id \in G$ is unique. □

This property of uniqueness allows us to speak of only one unique identity transformation, id, that maps an element to itself.

Theorem 2 (Cancellation Property of a Group)**.** *Let G be a group and $a, b, c \in G$. If $a \cdot b = a \cdot c$, then $b = c$.*

Proof. Let G be a group and $a, b, c \in G$. Assume $a \cdot b = a \cdot c$. Then, by multiplying both sides of the equation by a^{-1} on the left, we get

$$b = a^{-1} \cdot a \cdot b = a^{-1} \cdot a \cdot c = c$$

Therefore, $b = c$. □

See that cancellation property allows us to cancel the same elements on both sides of the equation, and thus it is essential for manipulating equations.

Now, hypothetically one element could have more than one inverse that would satisfy the definition of a group, however, the next theorem shows that is not the case.

Theorem 3 (Uniqueness of Inverses of a Group). *Let G be a group and $a \in G$. Then a^{-1} is unique in G.*

Proof. Let a^{-1} and b^{-1} be two inverses of a in G. Then using definition of a group $a^{-1}a = b^{-1}a = id$, where id is the identity element of G. Thus, (using cancelation property), $a^{-1} = b^{-1}$. □

Theorem 4 (Socks-Shoes Property). *For group elements a and b, $(ab)^{-1} = b^{-1}a^{-1}$.*

Proof. Let G be a group and $a, b \in G$. Then, (recall that from definition of a group $(ab)(ab)^{-1} = id$)

$$\begin{aligned}(ab)(b^{-1}a^{-1}) &= a(bb^{-1})a^{-1} \\ &= a \cdot id \cdot a^{-1} = aa^{-1} = id \\ &= (ab)(ab)^{-1}.\end{aligned}$$

Using the left cancelation property, we get

$$\begin{aligned}(ab)(b^{-1}a^{-1}) &= (ab)(ab)^{-1}. \\ (b^{-1}a^{-1}) &= (ab)^{-1}.\end{aligned}$$

□

We will use only the definitions and theorems listed in this section in the following proofs.

Now, let us go back to the permutations. The dihedral group D_3 represents the symmetries of an equilateral triangle. It has six elements: three rotations and three reflections. Is there a more compact way to represent the permutation of its vertices, as opposed to the mappings that we used in Definition 14?

Cycle notation offers a compact and efficient method for expressing permutations. It represents permutations as a product of disjoint cycles (without intersections), where each cycle is a sequence of elements that are cyclically permuted via transformation. This notation is particularly helpful when dealing with symmetric groups and investigating their properties.

Here are the same six elements/transformations of D_3 represented in cycle notation:

- The identity permutation (no change / 0 degree rotation): (1)(2)(3)
- Rotation by 120 degrees counterclockwise: (1 2 3)
- Rotation by 240 degrees counterclockwise: (1 3 2)
- Reflection across the axis passing through vertex 1: (1) (2 3)

- Reflection across the axis passing through vertex 2: (2) (1 3)
- Reflection across the axis passing through vertex 3: (3) (1 2)

For the sake of illustration, let us take the graph in Fig. 3.3, and apply the transformation of this graph using the reflection across the axis passing through vertex 3. What happens is this:

$$1 \mapsto 2$$
$$2 \mapsto 1$$
$$3 \mapsto 3$$

Thus, the order of vertices is the same as in the transform automorphism of graph in Fig. 3.4.

If we apply the same reflection again, we are back again with the initial order of vertices in the graph because of the cycle (1 2), i.e., by the first reflection 1 mapped to 2, by the second reflection 2 mapped back to 1.

Now, we want to understand when two groups are structurally the same. For this goal, we will define what is an isomorphism.

Definition 16 (Group isomorphism)**.** A ***group isomorphism*** ϕ from a group G to a group $\bar{G}$ is a bijective mapping (or function) from G onto $\bar{G}$ that preserves the group operation. That is,

$$\phi(ab) = \phi(a)\phi(b) \quad \text{for all } a, b \text{ in } G.$$

If there is an isomorphism from G onto $\bar{G}$, we say that G and $\bar{G}$ are isomorphic and write $G \approx \bar{G}$.

Definition 17 (Symmetric Group)**.** Let $A = \{1, 2, \ldots, n\}$. The set of all permutations of A is called the ***symmetric group*** of degree n and is denoted by $Sym(n)$. Elements of $Sym(n)$ can be phrased in the form of a mapping α where each element from A is permuted via a permutation function α:

$$1 \mapsto \alpha(1)$$
$$2 \mapsto \alpha(2)$$
$$3 \mapsto \alpha(3)$$
$$\vdots$$
$$n \mapsto \alpha(n)$$

It is easy to compute the order of $Sym(n)$. There are n choices of $\alpha(1)$. Once $\alpha(1)$ has been determined, there are $n-1$ possibilities for $\alpha(2)$ (since α is an injective mapping, we must have $\alpha(1) \neq \alpha(2)$). After choosing $\alpha(2)$, there are exactly $n-2$ possibilities for

$\alpha(3)$. Continuing along in this fashion, we see that $Sym(n)$ has $n(n-1)\cdots 3\cdot 2\cdot 1 = n!$ elements.

The symmetric groups are rich in subgroups, where ***subgroups*** are subsets of transformations of a group that again satisfy the definition of a group. The group $Sym(4)$ has 30 subgroups, and $Sym(5)$ has well over 100 subgroups.

Theorem 5 (Cayley's Theorem)**.** *Let G be a group. We show that G is isomorphic to a group of permutations (and subgroup of the symmetric group $Sym(n)$).*

Proof. To prove this theorem, we divide the work into these steps:

1. Define the mapping ϕ_a: Let G be any group. We must find a group $\bar{G}$ of permutations that we believe is isomorphic to G. Since G is all we have to work with, we will have to use it to construct $\bar{G}$. For any a in G, define a function ϕ_a from G to G by
$$\phi_a(x) = ax \quad \text{for all } x \text{ in } G.$$
(In words, ϕ_a is just multiplication by some $a \in G$ on the left.)

2. Prove that ϕ_a is a permutation on the set of element of G: $\phi_a(x) = \phi_a(y)$ iff $ax = ay$ or $x = y$. This shows that ϕ_a is an injective function. Let $y \in G$. Then $\phi_a\left(a^{-1}y\right) = (aa^{-1})y = y$, so that ϕ_a is onto. Thus, ϕ_a is a bijection from the set G onto itself, i.e., permutation.

3. Define the representation of G: Now, let $\bar{G} = \left\{\phi_a \mid a{\in}G\right\}$. Further, we want to prove that $\bar{G}$ is a group.

4. Observe the properties of ϕ_a to prove that $\bar{G}$ is a group: To verify this, we need to show that $\bar{G}$ satisfies all the properties of a group. Its elements are functions, thus we need to show that function composition holds. We first observe that for any a and b in G we have $\phi_a\phi_b(x) = \phi_a\left(\phi_b(x)\right) = \phi_a(bx) = a(bx) = (ab)x = \phi_{ab}(x)$, so that $\phi_a\phi_b = \phi_{ab}$. From this it follows that ϕ_{id} is the identity since $\phi_{id}(x) = id \cdot x = x$ and $(\phi_a)^{-1} = \phi_{a^{-1}}$ since $\phi_a \circ (\phi_a)^{-1} = \phi_{id} = \phi_{aa^{-1}} = aa^{-1}(x) = a(a^{-1}(x)) = \phi_a \circ \phi_{a^{-1}}$ and cancel ϕ_a on both sides. Since function composition is associative by definition, we have verified all the conditions for $\bar{G}$ to be a group. $\bar{G}$ is a group under the operation of function composition, called the ***left regular representation of*** G.

5. Show that G is isomorphic to a group of permutations $\bar{G}$: The isomorphism ϕ between G and $\bar{G}$ is now ready-made. For every a in G, define $\phi(a) = \phi_a$. If $\phi_a = \phi_b$, then $\phi_a(id) = \phi_b(id)$ or $a \cdot id = b \cdot id$. Thus, $a = b$ and ϕ is injective function. By the way $\bar{G}$ was constructed, we see that ϕ is onto. The only condition

that remains to be checked is that ϕ is operation-preserving. To this end, let a and b belong to G. Then

$$\phi(ab) = \phi_{ab} = \phi_a\phi_b = \phi(a)\phi(b).$$

6. Show that the group of permutations $\bar{G}$ is a subgroup of the symmetric group $Sym(n)$: If G is a finite group with n elements, then the left regular representation of G is a subgroup of the symmetric group $Sym(n)$, as each permutation in $\bar{G}$ is a bijection on the set G so it satisfies the definition of a symmetric group. Consequently, Cayley's Theorem indicates that every finite group of order n can be considered as a subgroup of the symmetric group of degree n. That is why we usually denote the symmetric group of any group set X simply as $Sym(X)$.

□

Cayley's Theorem is important for two contrasting reasons. One is that it allows us to represent an abstract group in the area of permutations that is much more tangible. Secondly, it demonstrates that the contemporary set of axioms selected for a group accurately abstracts its earlier permutation group counterpart developed by Galois. Indeed, Cayley's Theorem tells us that abstract groups are not different from permutation groups (Gallian, 2021).

We started this chapter with the promise to study symmetries on graphs. Is there a connection between graphs and groups? Indeed, the next theorem shows that there is a deep connection.

Theorem 6 (Frucht's theorem)**.** *For any finite group G there exists a graph Γ such that $Aut(\Gamma) \cong G$*

This is essential because not only groups (global symmetries) can be studied using permutations, but also using graph automorphisms without destroying the underlying structure. We do not provide a procedure to generate a graph for each group, since that would be beyond the scope of this thesis.

3.3 Partial symmetry and inverse monoids

In this section, we formalize the notion of partial symmetry in the mathematically precise framework of inverse monoids. We follow the classical textbook of Lawson (1998) and papers of Jajcay et al. (2021) and Jajcayova (2022). The primary motivation is to contrast and parallel the group theory with the inverse monoid theory. Note that sometimes one can see inverse semigroups being referred to as the formalization of partial symmetry. However, inverse monoids are just inverse semigroups with identity property (without any huge structural change), thus, for many purposes, these terms are interchanged.

Preliminaries

Definition 18 (Partial function (mapping))**.** Let A and B be any two sets. A ***partial function*** f from A to B, $f : A \rightharpoonup B$, is a function from a subset of A to a subset of B. The subset of A consisting of all those elements $a \in A$ for which $f(a)$ is defined is called the domain (of definition) of f, which we denote by $\operatorname{dom} f$. The image / range of f is the subset $ran(f) = f(\operatorname{dom}(f))$ of B. See Fig. 3.7.

Two special classes of partial functions are particularly important. For any two sets A and B there is a unique empty partial function from A to B which we denote by $id_\emptyset$. For every subset A_1 of A the identity function on A, denoted id_{A_1}, is a partial function from A to itself. Such partial functions are termed partial identities.

Definition 19 (Composition of Partial Functions)**.** Let f be a partial function from A to B and g a partial function from B to C. Then the ***composition of partial functions*** is a partial function $g \circ f$ from A to C, where the domain of $g \circ f$ is given by

$$\operatorname{dom}(g \circ f) = f^{-1}(\operatorname{dom} g \cap \operatorname{im} f)$$

and if $x \in \operatorname{dom}(g \circ f)$ then $(g \circ f)(x) = g(f(x))$. The image of $g \circ f$ is $g(\operatorname{dom} g \cap \operatorname{im} f)$. The case where $\operatorname{dom} g$ and $\operatorname{im} f$ have empty intersection causes no problems: $g \circ f$ is just the empty function. We usually write gf rather than $g \circ f$.

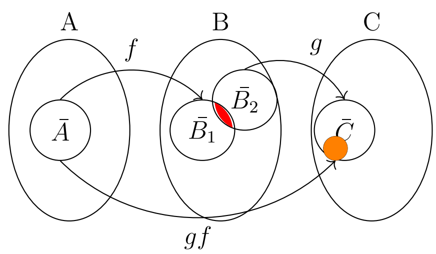

Figure 3.7: Composition of partial functions. As you can see in the picture, fg composition will only map to a subset of $\bar{C}$ in orange despite the fact that $\bar{B}_2$ maps to the whole subset $\bar{C}$. This is because the range of f overlap with g only in the small subset in red, thus their composition will only map this small subset in red to the one in orange. Note that both of these can be empty.

Note that the important message is that (composition of) partial functions may not be defined for all elements in the domain. Thence the use of the words ***locality*** and ***partiality***. If (the composition is) defined for all elements, it becomes a total function that we defined in the previous section.

We shall concentrate on those partial functions which induce bijections between their domains and images; we call such partial functions partial bijections. All partial identities and empty functions are partial bijections. If f is a partial bijection from A to B then we denote by f^{-1} the partial bijection from B to A which is the inverse of f. Thus, the domain of f^{-1} is $ran(f)$ and its range is $\mathrm{dom}(f)$. The composition of partial bijections is again a partial bijection. Some important properties of partial bijections are described below.

Definition 20 (Partial bijection)**.** Given sets A and B, and subsets $A' \subseteq A$ and $B' \subseteq B$, a ***partial bijection*** $f : A \rightharpoonup B$ satisfies the following two conditions:

1. ***Partial injection***: For every $a_1, a_2 \in A'$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$.
2. ***Partial surjection***: For every $b \in B'$, there exists $a \in A'$ such that $f(a) = b$.

In other words, a partial bijection is a partial mapping from a set A to a set B, $f : A \rightharpoonup B$ with bijective properties. Or, equivalently, it is a mapping from a subset of a set A to a subset of a set B, $f : A' \to B'$, with a bijective one-to-one correspondence between the elements of the subsets. The common size $|\,\mathrm{dom}(f)| = |\,\mathrm{ran}(f)|$ of the $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$ is called the ***rank*** of f.

The following proposition lists several essential properties of partial bijections.

Proposition 1 (Properties of partial bijections)**.** *Let A, B and C be sets, and let $f : A \rightharpoonup B$ be a partial bijection. Then:*

1. $f^{-1}f = id_{\mathrm{dom}\, f}$, a partial identity on A, and $ff^{-1} = id_{\mathrm{ran} f}$, a partial identity on B.
2. For a partial bijection $g : B \rightharpoonup A$, the equations $f = fgf$ and $g = gfg$ hold if, and only if, $g = f^{-1}$.
3. $\left(f^{-1}\right)^{-1} = f$
4. $id_{A_1} id_{A_2} = id_{A_1 \cap A_2} = id_{A_2} id_{A_1}$ for all partial identities id_{A_1} and id_{A_2} where $A_1, A_2 \subseteq A$.
5. $(gf)^{-1} = f^{-1}g^{-1}$ for any partial bijection $g : B \rightharpoonup C$.

Inverse monoid theory

Finally, we are ready to review the results of inverse monoid theory which is a representation of the concept of partial symmetry.

Definition 21 (inverse monoid)**.** An ***inverse monoid*** is a set S together with a binary operation $\cdot$ that satisfies four axioms: closure, associativity, generalized inverse, and identity. That is, for any $a, b, c \in S$, we have:

1. Closure: $a \cdot b \in S$
2. Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. *Generalized Inverse*: For each $a \in S$, there exists an element $a^{-1} \in S$ such that $a^{-1} \cdot a \cdot a^{-1} = a^{-1}$
4. Identity: For each $a \in S$, there exists an element $id \in S$ such that $id \cdot a = a \cdot id = a$

The main difference between groups and inverse monoids is in the nature of the inverse elements. Groups capture global symmetry, as applying the inverse operation undoes the original operation for the entire set ($aa^{-1} = id$). However, in inverse monoids, the inverses ($aa^{-1}a = a$) only partially undo the original operation, which is why the structure of inverse monoids is said to capture local symmetry.

Theorem 7 (Every group is inverse monoid)**.**

Proof. See that only thing that has to be proved is that inverse of group satisfies the generalized inverse of inverse monoids, other criteria in the definitions of group and inverse monoid are the same.

Let S be any inverse monoid. For each $a \in S$, there exists an element $a^{-1} \in S$ such that $a^{-1} \cdot a \cdot a^{-1} = a^{-1}$. Now substitute $a^{-1} \cdot a$ for the identity from the Definition 14 of the group. $id \cdot a^{-1} = a^{-1}$, and this is equal to $a^{-1} = a^{-1}$.

□

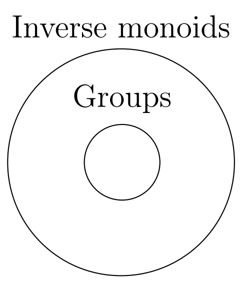

Figure 3.8: The concept of group is a subconcept of inverse monoid.

From now on, it should be clear that the concept of partial symmetry is not in opposition to the concept of symmetry, but it is rather its generalization, and so the concept of partial symmetry encompasses the concept of global symmetry which can be nicely illustrated using Venn's diagrams in Fig. 3.8. Despite this closeness between the concepts, there are also structures such as those in Fig. 3.9 which do not have any of the global symmetries, but if you delete just one node, all of its induced subgraphs have global symmetries.

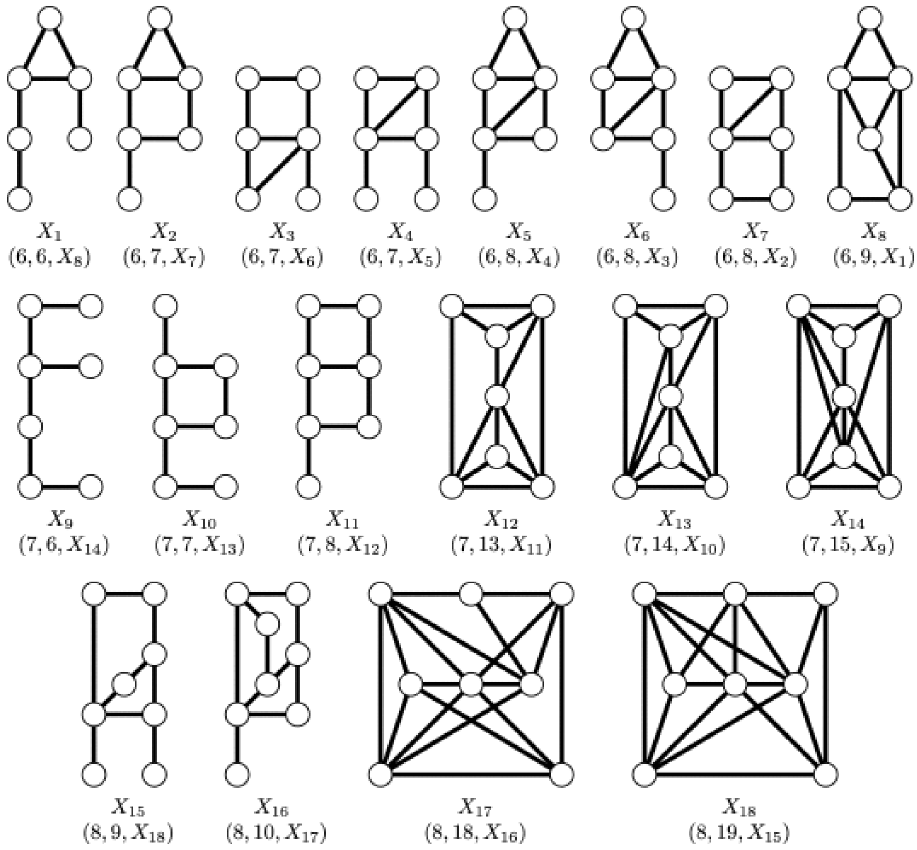

Figure 3.9: These graphs are ***minimally asymmetric***, i.e., there is a single vertex that you can delete, and you will get a structure that does not have any induced subgraph that is asymmetric (Schweitzer, 2017).

Now let us define natural ***partial order*** on the inverse monoids.

Definition 22. Let $s, t \in S$, then, $s \leq t <=> s = te$ for some idempotent e, where ***idempotency*** is this property: $ee = e$. Idempotency is a kind of redundancy that does not change anything if applied multiple times in a consecutive sequence.

Lemma 1. *Let S be an inverse monoid. Then the following are equivalent:*

1. $s \leq t$.
2. $s = ft$ *for some idempotent* f.
3. $s^{-1} \leq t^{-1}$.
4. $s = ss^{-1}t$.
5. $s = ts^{-1}s$.

Proposition 2. *Let S be an inverse monoid, then the relation $\leq$ is a partial order on S.*

This proposition shows that our definition of partial order on inverse monoids is indeed partial order, as we would want it to be.

Proof. We need to prove that $\leq$ is a reflexive, antisymmetric, and transitive operation on S.

Since $s \leq s$ is (according to (4/5) of Lemma 1) equal to $s = s(s^{-1}s)$, and so $s = s$ because of the inverse property of semigroups, the relation is reflexive. Now, we show that $\leq$ is antisymmetric. Let $s \leq t$ and $t \leq s$ Then, using (5) in Lemma 1, we have $s = ts^{-1}s$ and $t = st^{-1}t$. So,

$s = ts^{-1}s$	Initial equation using (5) in Lemma 1 for s
$= st^{-1}ts^{-1}s$	Using the same Lemma for t.
$= tt^{-1}ss^{-1}s$	Subtituting $st^{-1}t$ for $tt^{-1}s$ (Lemma 1 (4) applied for t.)
$= tt^{-1}s$	using inverse property
$= t$	Using (4) in Lemma 1

Lastly, let us show that it is transitive. Let $s \leq t$ and $t \leq u$. We have to show from these that $s \leq u$. By definition of a partial order, $s = te$ and $t = uf$. Make a substitution of the second equation into the first one, and we get $s = u(fe)$ where both f and e are idempotents (and composition of idempotents is idempotent), therefore, $s \leq u$. □

This proposition is crucial: if an inverse monoid that has an equivalence relation instead of a partial order, it turns into a group (Lawson, 1998).

Now, let us focus on what makes two inverse monoids structurally the same.

Definition 23 (Injective homomorphism)**.** An ***homomorphism*** ϕ from an inverse monoid S to an inverse monoid $\bar{S}$ is a mapping (or function) from S onto $\bar{S}$ that preserves the inverse monoid operation. That is,

$$\phi(ab) = \phi(a)\phi(b) \quad \text{for all } a, b \text{ in } S.$$

If there is a homomorphism from S onto $\bar{S}$, we say that S and $\bar{S}$ are homomorphic. ***Injective homomorphism*** is an injective mapping from S onto $\bar{S}$ that preserves the inverse monoid operation.

See that injective homomorphism is very similar to isomorphism used in group theory, but as opposed to isomorphism, injective homomorphism is not surjective, thus some elements in the range might not have corresponding elements in the domain.

Definition 24 (Symmetric inverse monoid $PSym(X)$)**.** The collection of all partial bijections from a set to itself forms a ***symmetric inverse monoid on X***, usually denoted as $PSym(X)$.

Theorem 8 (Wagner-Preston representation theorem)**.** *Let S be an inverse monoid. Then there is a set X and an injective homomorphism $\Theta \colon S \to PSym(X)$ such that $a \leq b \implies \Theta(a) \subseteq \Theta(b)$, where $a, b \in S$.*

Proof. Note that this theorem is an analogue to the Cayley's theorem for group theory. We provide the outline of the proof so that the reader can contrast it to Cayley's theorem.

1. Define the mapping Θ_a.

2. Observe its (bijective) properties.

3. Define the (representational) mapping $\Theta \colon S \to PSym(X)$.

4. Show that Θ is a homomorphism.

5. Show that Θ is injective.

□

This theorem demonstrates that every inverse monoid can be embedded into a symmetric inverse monoid (of partial permutations), providing a powerful tool to study inverse monoids through the lens of partial bijections on some set X.

Further, recall cycle notation in the group theory – A cycle represents a permutation that cyclically rearranges all the elements within its domain. However, it is essential to differentiate this notion in partial permutations. The cycle notation of partial permutations is a generalization and allows also a ***path***, which (unlike a cycle) ends when it reaches the undefined element: $\operatorname{dom}(x_k, \dots x_2, x_1] = \{x_{k-1}, \dots, x_2, x_1\}$ and $\operatorname{ran}(x_k, \dots x_2, x_1] = \{x_k, \dots, x_3, x_2\}$ (Jajcayova, 2022). Every partial permutation in PSym(X) is the union of pairwise disjoint (non-overlapping) paths and cycles. For example, consider a partial permutation on the set $1, 2, 3, 4$ that maps 1 to 3, 3 to 4, and 2 to 2. This partial permutation can be uniquely written as (1 3 4] (2) (Jajcay et al., 2021).

As an example, we will show the application of inverse monoids and partial symmetries to graphs.

Firstly, note that all partial automorphism of any graph Γ are partial permutations of its vertices $V(\Gamma)$, therefore $PAut(\Gamma)$ actually belongs to $PSym(V(\Gamma))$, $PAut(\Gamma) \subseteq PSym(V(\Gamma))$, since it satisfies the definition of inverse monoid.

In examining the structure of inverse monoids, four key equivalence relations are essential. These are called ***Green's relations***. Let φ_1 and φ_2 be partial permutations. Then Green's relations are:

$$\varphi_1 \mathcal{L} \varphi_2 \Leftrightarrow \operatorname{dom} \varphi_1 = dom \varphi_2$$

$$\varphi_1 \mathcal{R} \varphi_2 \Leftrightarrow \operatorname{ran} \varphi_1 = \operatorname{ran} \varphi_2$$

$$\varphi_1 \mathcal{H} \varphi_2 \Leftrightarrow \varphi_1 \mathcal{L} \varphi_2 \text{ and } \varphi_1 \mathcal{R} \varphi_2$$

Thus, we can say that $\mathcal{H}$ is an intersection of $\mathcal{L}$ and $\mathcal{R}$ classes.

$$\varphi_1 \mathcal{D} \varphi_2 \Leftrightarrow \varphi_1 \text{ and } \varphi_2 \text{ have the same rank.}$$

That is why we can say that $\mathcal{D}$ is a disjoint union of $\mathcal{L}$, and $\mathcal{R}$ classes where both φ_1 and φ_2 are partial permutations $\in PSym(X)$ for some set X.

Each $\mathcal{R}$ class and each $\mathcal{L}$-class contain precisely one idempotent, and the $\mathcal{H}$-classes containing these idempotents are the maximal/the biggest subgroups of the inverse monoid since they both preserve domains and ranges.

In simpler words,

1. $\mathcal{L}$-class relation relates two partial permutations iff they have the same domain (the set of vertices they act on).

2. $\mathcal{R}$-class relation relates two partial permutations iff they have the same range (the set of vertices they map to).

3. $\mathcal{H}$-class relation relates two partial permutations iff they have both the same range and domain.

4. $\mathcal{D}$-class relation relates two partial permutations iff φ_1 and φ_2 have the same rank.

Let us give an example from Jajcay et al. (2021). The $\mathcal{R}$-class of the partial automorphism $(1) \vee (2)$ consists of all partial automorphisms of range $\{1, 2\}$, that is, of $\{(1) \vee (2), (12), [123), (2) \vee [13)\}$. Similarly, the $\mathcal{L}$-class of $\lfloor 1) \vee (2)$ consists of all partial automorphisms with domain $\{1, 2\}$, that is, of $\{(1) \vee (2), (12), [321), (2) \vee [31)\}$. Note that the $\mathcal{L}$-class of $(1) \vee (2)$ contains exactly the inverses of the elements in its $\mathcal{R}$-class. This is no coincidence: In any inverse monoid, $a\mathcal{R}b$ iff $a^{-1}\mathcal{L}b^{-1}$.

The $\mathcal{H}$-class of (1) (2) in PAut (Γ_0) is $\{(1) \vee (2), (12)\}$. Note that it forms a subgroup.

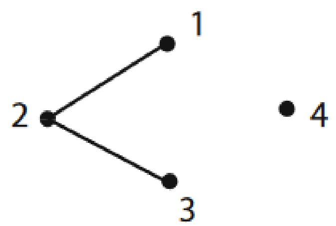

Figure 3.10: Example of a graph Γ_0 with four vertices.

For the $\mathcal{D}$-class consider graph Γ_0 in Fig. 3.10 and the partial automorphisms (12), $(34) \in \mathrm{PAut}(\Gamma_0)$ again. Both have rank 2, but there is no partial graph automorphism $\psi \in \mathrm{PAut}(\Gamma_0)$ with $\mathrm{dom}\,\psi = \{1, 2\}$ and $\mathrm{ran}\,\psi = \{3, 4\}$, as $\{1, 2\}$ is an edge while $\{3, 4\}$ is not. As it turns out in partial automorphism monoids, the $\mathcal{D}$ relation corresponds to isomorphism classes of induced subgraphs, as formulated in the following propositions which answer what is the structure of $PAut(\Gamma)$ for some graph Γ. The details are provided in Jajcay et al. (2021).

Proposition 3 (Jajcay et al. (2021)). *For any graph Γ, the D-classes of $PAut(\Gamma)$ correspond to the isomorphism classes of induced subgraphs of Γ, that is, two elements are $\mathcal{D}$-related iff the subgraphs induced by their respective domains (or ranges) are isomorphic.*

This means, that the partial order for $\mathcal{D}$-classes corresponds to induced subgraph relation. This fact is used heavily when finding inverse monoids of partial automorphisms for particular graphs. It also indicates that computationally this problem is hard, as we have to go through all induced subgraphs of a given graph.

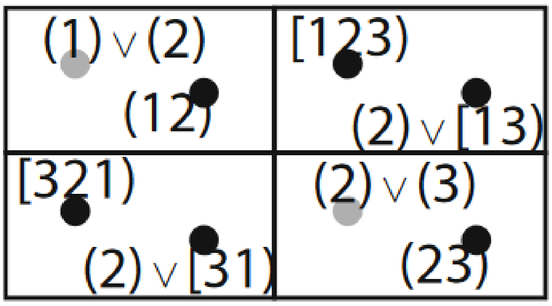

Figure 3.11: Egg-box diagram of the D-class of the edges in $PAut(\Gamma_0)$ corresponding to graph in Fig. 3.10. After Jajcay et al. (2021).

In an inverse monoid, each $\mathcal{D}$-class is a disjoint union of $\mathcal{R}$-classes, and a disjoint union of $\mathcal{L}$-classes, and any $\mathcal{R}$ - and $\mathcal{L}$-class of a $\mathcal{D}$-class intersect in an $\mathcal{H}$-class. Moreover, every $\mathcal{D}$-class contains the same number of $\mathcal{R}$-classes as $\mathcal{L}$-classes. A $\mathcal{D}$-class is therefore usually depicted in what is called an ***egg-box diagram***: The whole rows are the $\mathcal{R}$-classes, the whole columns are the $\mathcal{L}$-classes, the small rectangles are the $\mathcal{H}$-classes. These egg-boxes are arranged in such a way that the $\mathcal{H}$-classes containing idempotents are on the main diagonal.

For an example, see Fig. 3.11 which depicts the $\mathcal{D}$-class of PAut (Γ_0) corresponding to the edges of the graph. The first row (column) corresponds to the range (domain) $\{1,2\}$, and the second row (column) corresponds to the range (domain) $\{2,3\}$. These belong to the same $\mathcal{D}$-class since both $\{1,2\}$ and $\{2,3\}$ represent edges of PAut (Γ_0), and therefore, there exists a partial automorphism of rank 2 mapping $\{1,2\}$ to $\{2,3\}$. The idempotents are colored gray.

In the case of finite inverse monoids, the $\mathcal{D}$-classes form a partially ordered set: If we denote the $\mathcal{D}$-class of a by D_a, we put $D_a \leq D_b$ if $a = xby$ for some $x, y \in \mathcal{S}$, that is, if b can be multiplied into a. In the case of PSym(X), this is just the ordering of $\mathcal{D}$-classes according to their rank so that the $\mathcal{D}$-classes form a chain of egg-box diagrams visible in Fig. 3.12. The minimum element is the $\mathcal{D}$-class of partial permutations of rank 0, which is just $\{\mathrm{id}_\emptyset\}$. The maximum element is the $\mathcal{D}$-class of rank $|X|$ partial permutations, which are the permutations of X. This maximal $\mathcal{D}$-class, therefore, consists of a single $\mathcal{H}$-class, which is the symmetric group Sym(X).

In the case of partial automorphism monoids, this partial order corresponds to the induced subgraph relation between the isomorphism classes of graphs, as implied by the following proposition.

Finally, we are ready to answer the question formulated in Jajcayova (2021): "When is an inverse monoid of partial permutations the partial automorphism monoid of a graph?"

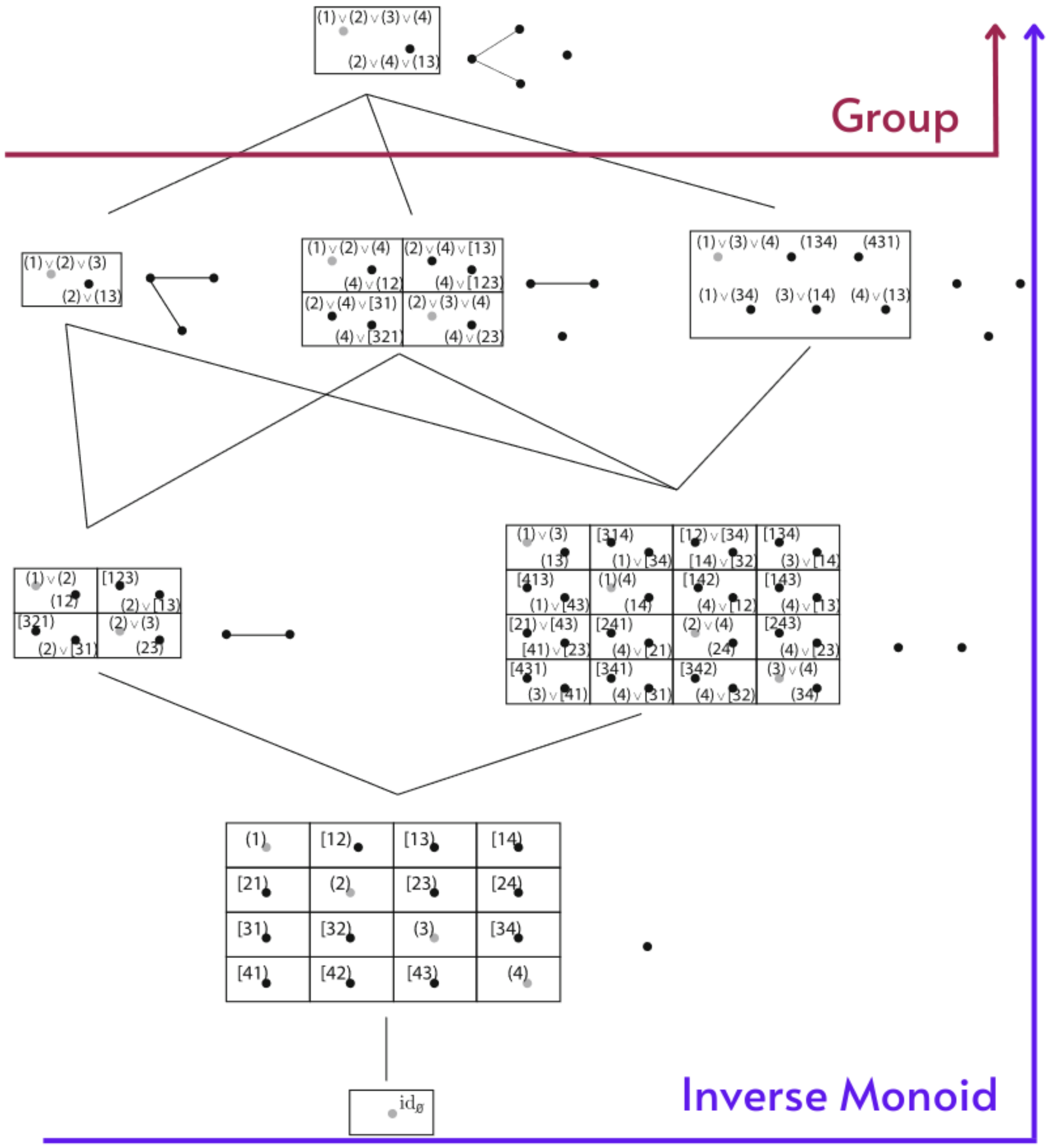

Figure 3.12: Example of partial symmetries of a graph (partial graph automorphisms that satisfy the conditions for inverse monoid that admits graph) represented by inverse monoids, and contrast to what is captured by the group theory. Figure adapted from Jajcay et al. (2021).

Theorem 9 (Jajcay et al. (2021)). *Given an inverse submonoid $S \leq P\,\mathrm{Sym}(X)$, where X is a finite set, there exists a graph with vertex set X whose partial automorphism monoid is S iff the following conditions hold:*

1. S is a full inverse submonoid of PSym(X),
2. for any compatible subset $A \subseteq S$ of rank 1 partial permutations, if S contains the join of any two elements of A, then S contains the join of the set A,
3. the rank 2 elements of S form at most two D-classes,
4. the $\mathcal{H}$-classes of rank 2 elements are nontrivial.

Moreover, the next theorem describes "when is an inverse monoid isomorphic to the partial automorphism monoid of a graph" (Jajcayova, 2021)?

Theorem 10 (Jajcay et al. (2021)). *Given a finite inverse monoid S, there exists a finite graph whose partial automorphism monoid is isomorphic to S iff the following conditions hold:*

1. *S is Boolean,*
2. *S is fundamental,*
3. *for any subset $A \subseteq S$ of compatible 0-minimal elements, if all 2-element subsets of A have a join in S, then the set A has a join in S,*
4. *the 0-minimal elements of S are $\mathcal{D}$-equivalent,*
5. *S has at most two D-classes of height 2,*
6. *the $\mathcal{H}$-classes of the height $2\mathcal{D}$-classes of S are nontrivial.*

The partial automorphism monoid $P\mathrm{Aut}(\Gamma)$ of a graph Γ is a rich and complex structure, more so than the classical automorphism group $\mathrm{Aut}(\Gamma)$. The number of local identities in $P\mathrm{Aut}(\Gamma)$ is exponential with respect to the order of the graph.

Computing $P\mathrm{Aut}(\Gamma)$ for a graph Γ involves repeatedly computing automorphism groups in multiple stages. The classical $\mathrm{Aut}(\Gamma)$ must be found as a final top D-class, along with automorphism groups of all induced subgraphs. Generally, constructing the automorphism group is at least as difficult as solving the graph isomorphism problem. There are $|P\,\mathrm{Sym}(\{1,2,\ldots n\})| = \Sigma_{i=0}^{n}\binom{n}{i}^2 i!$ elements in $PAut(\Gamma)$. However, it can be done effectively for many graph classes.

3.3.1 Quantification of (partial) symmetry

In this section, we shortly propose how partial symmetry can be quantified. Since this question is not yet fully resolved, we will shortly list possible ways that first came to one's mind:

1. (top-down) approach would be to use a depth of partial symmetry, i.e., the minimal number of decomposition steps until we come to the underlying full non-trivial symmetry layers. This approach was already described in Erdős and Rényi (1963).

2. I also suggest (bottom-up) approach, i.e., define a distance function based on the minimal number of steps that one must take to create a non-trivial group of symmetries out of partial symmetry.

From the point of view of phenomenology, the second approach seems more suitable. Take, for instance, some slightly asymmetrical faces such as those in Fig. 1.5, then it would be more reasonable to count the number of changes that would 'repair' the full symmetry.

3.4 Contrast between symmetries and partial symmetries (groups and inverse monoids)

3.4.1 Formal differences and similarities

These are the main takeaways from the previous sections:

1. We have seen that the essential difference between the two theories is how they define the inverse property. Inverse monoids have inverses that allow substructures.

2. We have proved that group theory is a subtheory of inverse monoid theory, and thus these concepts are not in opposition, but rather in subconcept relation.

3. Interestingly, in some cases, such as in the minimally asymmetric graphs shown in Fig. 3.9, the transition from local symmetries to full global symmetries is literally just one vertex away.

4. Both Cayley's theorem and the Wagner-Preston representation theorem establish the existence of an isomorphism (or an injective homomorphism in the case of the Wagner-Preston theorem) between a given algebraic structure (group or inverse monoid) and a permutation structure (partial permutations or permutations). Cayley's theorem states that every group is isomorphic to a symmetric group of permutations, while the Wagner-Preston representation theorem states that every

inverse monoid can be embedded into a symmetric inverse monoid semigroup of partial permutations. Thus, both of these abstract concepts can be studied using easier combinatorial structures – (partial) permutations.

5. When we apply both of these concepts to graphs, partial symmetries of graphs are about finding induced subgraphs that have partial symmetries. An example and contrast of (partial) symmetries of a graph as represented by inverse monoids are provided in Fig. 3.12. It clearly shows that group theory is just the tip of the iceberg.

6. Computing partial symmetries is at least as hard as computing symmetries, and computing symmetries has an exponential difficulty (Jajcayova, 2022). Despite this, there are classes of graphs where computing partial symmetries is doable.

3.4.2 Partial symmetry, fractals, and complexity

In this section, our primary motivation is to further demonstrate the obvious restrictivity of the concept of global symmetry as opposed to local symmetry.

Sierpinski triangle

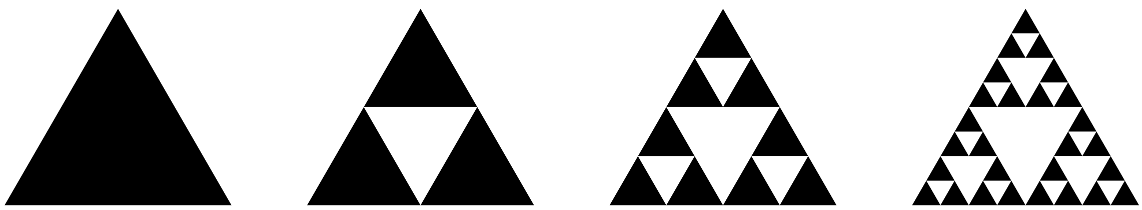

Figure 3.13: Generation of Sierpenski triangle for demonstration of simple fractal self-similarity that is not capturable by the concept of group.

Sierpinski triangle is a triangular iteratively generated fractal, shown in Fig. 3.13. To generate the Sierpinski triangle, start with a single equilateral triangle. Divide this triangle into four smaller equilateral triangles by connecting the midpoints of each side. Then, remove the central triangle formed by these midpoints. Repeat this process with each remaining smaller triangle, dividing it into four even smaller triangles and removing the central one. Continue this iteration infinitely many times, resulting in the fractal Sierpinski triangle. From the point of view of group theory, all of these steps of creating an infinite Sierpinski triangle are summarized by D_3 group, which can be covered by the graph shown in Fig. 3.3. The problem is that the concept of a group does not capture all the obvious wealth of its self-similar substructure. From the perspective of groups theory, all of these steps in Fig. 3.13 are the same D_3 group with the same number of global transformations and the identical graph representation, despite their rich recursive substructure which can be captured using inverse semigroups (Lawson, 1998).

Whether all the fractals could be captured by the theory of inverse semigroups or inverse monoids remains to be seen. Nevertheless, Lawson (1998) indicates that various kinds of fractal can be captured by inverse monoids much better than using the standard groups.

Penrose tilings

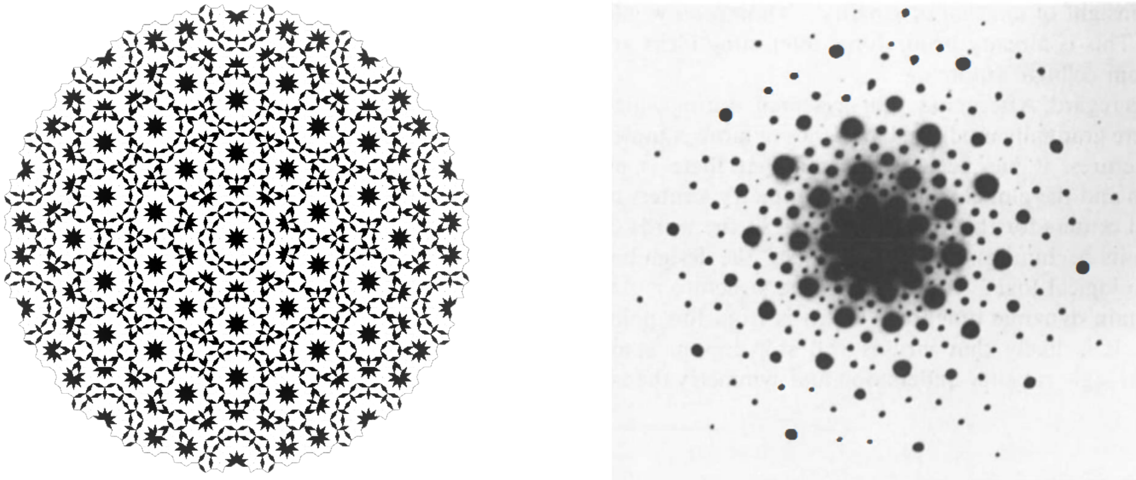

Figure 3.14: Penrose tiling that resembles quasicrystal shown in Fig. 4 in the Section 0.3. Generated using web application developed by Bhatia (2023).

Secondly, let us introduce ***Penrose tilings*** which can be understood as an instantiation of inverse semigroups precisely because they do not have some properties of the global symmetries (translational symmetry, and some do not have to have even rotational symmetry, and global inverses), they still have very rich underlying local symmetries, hierarchical structure and proportion as can be seen in Fig. 3.14.

In plain language, ***tilings*** are coverings of a plane with a set of tiles. We can see such tilings on the streets, where pavements are frequently tiled with hexagons or squares. Penrose tilings are a type of non-periodic tiling generated by an aperiodic set of prototiles (basic tiles from which you start) named after Sir Roger Penrose, who investigated these tilings in the 1970s (Penrose, 1974, 1979a, 1979b). The tilings are non-periodic because they lack translational symmetry, meaning that a shifted copy will never match the original exactly (minutephysics, 2022). As we will show, these non-periodic tilings lead toward the golden ratio, which is the definition of proportionality.

Tilings in general are interesting and useful for a variety of reasons, some being:

1. Combinatorics and counting: Tiling problems often involve counting the number of ways a particular shape can be arranged or combined to cover a region. This

engages the field of combinatorics, the study of counting and arranging objects, which has applications in many areas of mathematics, including probability theory, graph theory, and algebra.

2. Geometric properties: Tiling problems can help us understand and explore the geometric properties of shapes, such as their area, perimeter, symmetry, and tessellation patterns. Studying these properties can lead to discoveries about the relationships between different shapes and can inform the design of new geometric structures.

3. Optimization: Tiling problems often involve finding the most efficient or optimal way to cover a given area. This can be applied to real-world problems, such as optimizing resource allocation.

4. Algorithm development: Tiling problems frequently require the development of new algorithms and computational methods to solve them such as the one described in Section 3.4.2, which can then be applied to other problems in computer science, and related fields.

5. Recreational mathematics: Tiling problems have long been a popular topic in recreational mathematics, as they can be both visually appealing and intellectually challenging. They often involve elegant and intricate patterns, which can lead to the creation of puzzles, games, and artwork.

6. Connections to other disciplines: Tiling problems are connected to various branches of science, such as crystallography (the study of atomic and molecular structures), materials science (designing materials with specific properties), and even biology (understanding the arrangement of cells in tissues or the structure of virus capsids). Specifically, Penrose tilings have been successfully applied to study the quasicrystals, i.e., crystals whose structure does not have translational symmetry.

7. Aesthetic appeal: The study of tiling problems can lead to the creation of visually appealing designs and patterns. This has applications in architecture, design, and art, where an understanding of tiling patterns can inspire new creative works. For instance, Penrose argued that these non-periodic are appealing as opposed to simple periodic tilings (Penrose, 1974).

It has been a longstanding problem in mathematics for centuries whether one can cover the whole infinite plane without gaps and overlaps non-periodically, i.e., without a repetition, with some set of basic tiles. This problem of covering a plane (tessellation/tiling) is trivial (and boring) for periodic tilings using regular polygons such as squares, rectangles, hexagons, and triangles.

Despite this difficulty of non-periodic tilings, Penrose has managed to cover the whole plane non-periodically with just two basic tiles such as thin and thick rhombuses (see Section 3.4.2), since then the search for a single tile that can cover the plane began. In the words of Penrose - "It is not known whether there is a single shape that can tile the Euclidean plane only non-periodically. For the hyperbolic (Lobachevski) plane a single shape can be provided which, in a certain sense, tiles only non-periodically - but in another sense a periodicity (in one direction only) can occur" (Penrose, 1979b). What about the classical plane?

Surprisingly, as of lately, such a monotile has been found in Smith et al. (2023). They call this monotile ***Einstein's hat***.

De Bruijn method

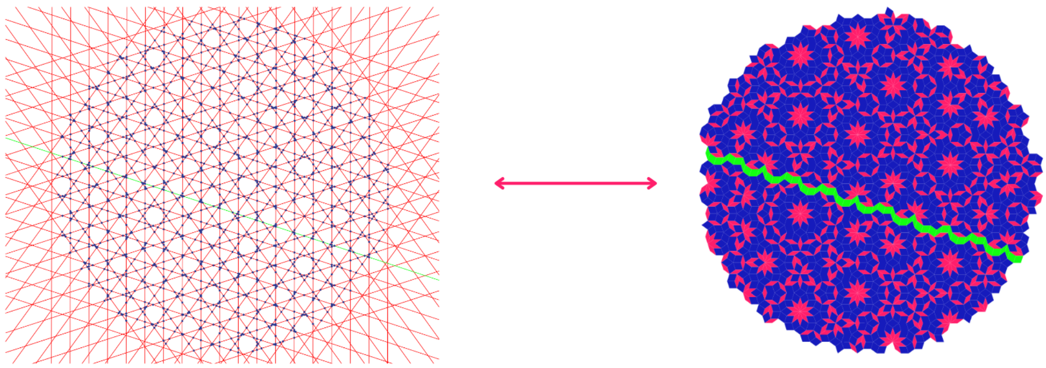

Figure 3.15: Process of generating pentagrid and quasi-periodic tiling using van de Bruijn method using web application programmed by Bhatia (2023). The ratio of thin and thick rhombuses approaches the golden ratio on each line, such as the one marked by green.

In this section, we follow de ***Bruijn multigrid method*** of generating tilings as described in Bhatia (2023).

1. Firstly, de Bruijn observed that every set of intersecting lines will generate a complete tiling. To see this, start with a set of lines (each set will correspond to ***1-fold***, i.e., to have 5-fold tilings start with 5 sets of parallel lines) and find the points where they intersect. See Fig. 3.16.

2. Secondly, at each intersection, draw equilateral polygons (tiles) whose sides are perpendicular to the lines. Notice that the shape of the tile will be determined by the angle of the intersecting lines. See Fig. 3.17.

3. Squeeze the tiles on each line and match them.

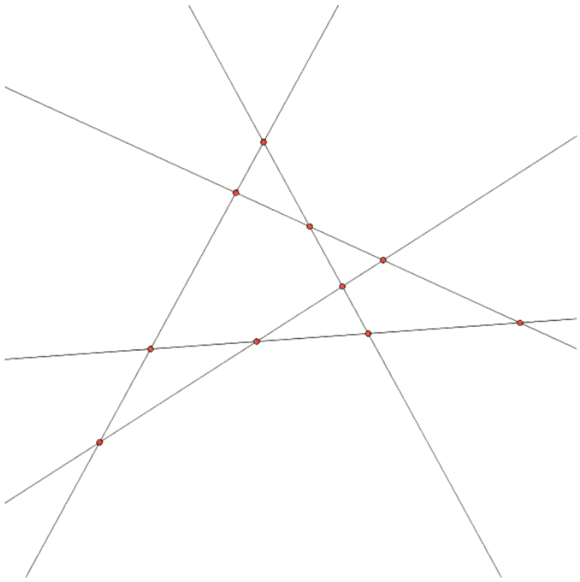

Figure 3.16: Any intersecting lines will form a tiling.

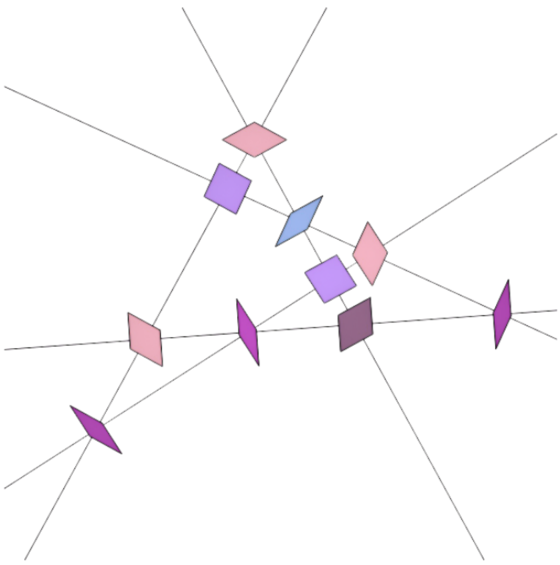

Figure 3.17: Any intersecting lines will form a tiling.

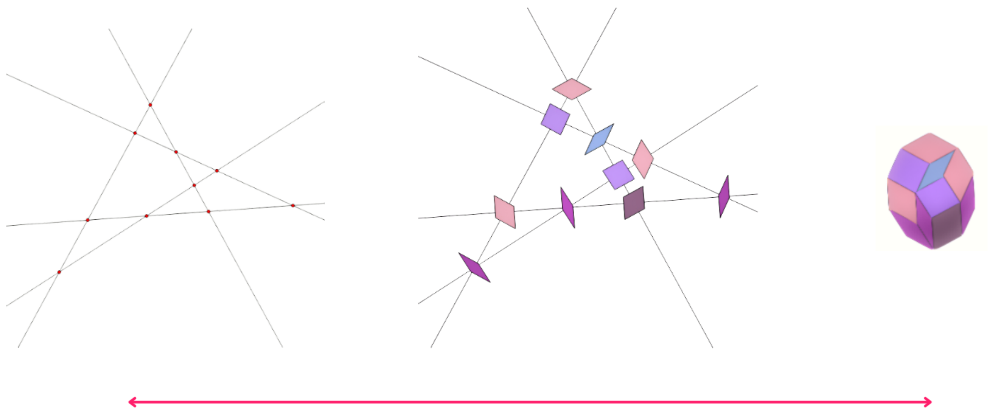

Figure 3.18: The whole process of constructing an arbitrary tiling of a plane.

The whole process is depicted in Fig. 3.18

To construct Penrose tilings, we need one additional condition, to perform the first step draw five collections of equally-spaced parallel lines so that each collection set is tilted $360°/5 = 72°$ from the previously drawn collection (obviously except the first collection). This grid of parallel lines is called ***pentagrid***.

Now, apply de Bruijn method, i.e. at each intersection draw equilateral polygons (tiles) whose sides are perpendicular to the lines. Since a set of lines in the pentagrid can intersect at two possible angles (72° and 36° and their corresponding remainders to the sum of 180°, see a zoom of Fig. 3.15 in Fig. 3.19), this pattern will have only two possible basic elemental tiles known as thin and thick rhombuses.

You can shift the lines around (since shift preserves the angle) and create infinitely many Penrose tilings, see one case in Fig. 3.15.

Proof of non-periodicity via Golden ratio ϕ

We provide intuitive arguments to believe the fact that Penrose tilings are non-periodic. The details of the proof are outside the scope and purpose of this thesis. The interested reader can find several references in the bibliography.

The whole idea behind the non-periodicity is that if Penrose tilings were periodic, then there would exist a certain period p after which the whole pattern would repeat itself.

If there would exist a certain period p after which the patterns would repeat, we would obviously expect after each period a stable ratio of different tiles out of which the tiling is made. However, the ratio of thin and thick rhombuses approaches the golden ratio on each line, such as the green line in Fig. 3.15). Since the golden ratio is an irrational number (by definition without decimal termination and repeatability), there is not any period of repetition.

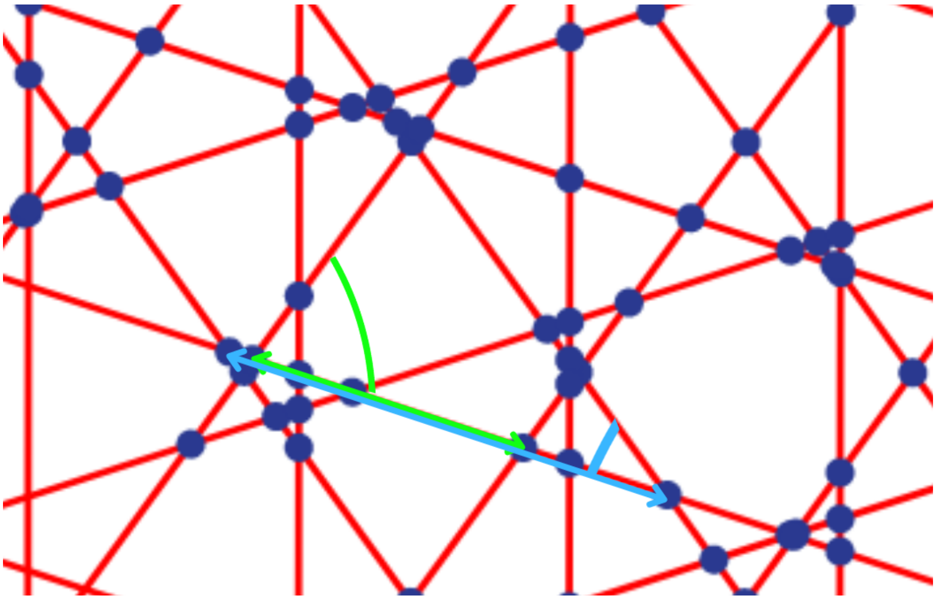

Figure 3.19: Zoomed in grid of Fig. 3.15. The only possible angles of intersection in pentagrid are 72° in green and 36° in blue (and their remainder to sum of 180°) with their corresponding distances between the intersections with 72° in green and 36° in blue. The green arrow line is the distance between subsequent intersections with 72° angles, and the blue line is the distance between subsequent intersections with 36° angles.

Let us explain why this ratio approaches the golden ratio. There are multiple methods how to show this.

The first method is related to the Bruijn method of construction that we utilized above. The idea is that since a set of lines in the pentagrid can intersect at only two possible angles (72°and 36°), it is obvious that thin rhombuses will be at the intersection with 36°, and thick rhombuses at the intersection with 72°. Given that the gap between each parallel line in a set is equal to 1, the ratio of the frequency of thin and thick rhombuses will correspond to the ratio of intersections with 36°, and intersections with 72°.

Let N_{36} represent the number of intersections with 36°, and N_{72} represent the number of intersections with 72°. The ratio of thin and thick rhombuses, x, can be represented as:

$$x = \frac{N_{36}}{N_{72}} \tag{3.1}$$

This ratio can be further reduced to the ratio of the distance between intersections with 36° and intersections with 72°. A little trigonometry, using the green and blue lines in Fig. 3.19 and perpendicular lines, will yield us:

$$x = \frac{N_{36}}{N_{72}} = \frac{\frac{1}{sin(36°)}}{\frac{1}{sin(72°)}} = \phi \tag{3.2}$$

However, this proves only that the ratio on one line is ϕ, in one direction. The whole proof is longer and more complicated, but this gives us an intuition of how it works. Therefore, let us rather explore another way of looking at this non-preriodicity.

The second method will utilize an idea recursion (function calls itself repeatedly until it reaches a base case). The tiling can be constructed by recursively applying substitution rules to the tiles (Penrose, 1979a, 1979b). For a tiling using rhombuses, the substitution rules can be defined as follows:

1. Subdivide a thick rhombus into two smaller thin and one smaller thick rhombuses.
2. Subdivide a thin rhombus into one smaller thick and one smaller thin rhombuses.

See Penrose (n.d.) for how this subdivision works visually.

Let $T(n)$ and $t(n)$ denote the number of thick and thin rhombuses at the nth iteration, respectively. As a correspondence to the substitution rules, we can write the following recurrence relations:

$$T(n+1) = 2t(n) + T(n)$$
$$t(n+1) = t(n) + T(n)$$

Now, let's find the ratio of thick and thin rhombuses at the $(n+1)$th iteration:

$$\frac{T(n+1)}{t(n+1)} = \frac{2t(n) + T(n)}{t(n) + T(n)}$$

To show that this ratio approaches the golden ratio, consider the limit of this ratio as n approaches infinity:

$$\lim_{n\to\infty} \frac{T(n+1)}{t(n+1)} = \lim_{n\to\infty} \frac{2t(n) + T(n)}{t(n) + T(n)}$$
$$= \lim_{n\to\infty} \frac{2\frac{t(n)}{T(n)} + 1}{\frac{t(n)}{T(n)} + 1}$$

Note that we just multiplied by 1, $\frac{1}{T(n)}$, both the numerator and denominator. Now, let $x = \lim_{n\to\infty} \frac{t(n)}{T(n)}$. As n approaches infinity, the ratio of consecutive terms in the sequence converges, so we have:

$$x = \frac{2x+1}{x+1}$$

Now multiply both sides by the denominator, $(x + 1)$, to eliminate the fractions:

$$x^2 + x = 2x + 1$$

This simplifies to:

$$x^2 - x - 1 = 0$$

The quadratic formula states that for a quadratic equation of the form $ax^2+bx+c = 0$, the solutions for x are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this case, $a = 1$, $b = -1$, and $c = -1$. Plugging these values into the quadratic formula, we get:

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

Simplifying this expression, we get:

$$x = \frac{1 \pm \sqrt{5}}{2}$$

This gives us two solutions for x: $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. The first solution, $\frac{1+\sqrt{5}}{2}$, is known as the ***golden ratio*** and is commonly denoted by the Greek letter phi (ϕ). The first solution is the one that we were looking for, since it is the only positive solution (the ratio of those rhombuses must be positive) to this equation:

$$x = \frac{1 + \sqrt{5}}{2} \approx 1.6180339887$$

This shows that the ratio of thin and thick rhombuses in Penrose tilings approaches the golden ratio as the tiling is iteratively inflated/deflated. Indeed, if one would count the ratio of those rhombuses on each line generated by de Bruijn method (see the green line on Fig. 3.15), the ratio of thin and thick rhombuses approaches the golden ratio.

Golden ratio and Fibonacci numbers

I feel obliged to leave a few comments on the nature of the Golden ratio since it occurs in many examples that we provided in this thesis. Originally, Penrose started where Kepler ended – by considering the impossibility of covering the plane with regular pentagons, whose diagonals have golden ratio if sides are of the length of 1 such as the one in Fig. 3.20. He then observed that if one would try to cover the plane with pentagons, there would be holes that one would need to cover with different tiles from the pentagon such as half stars, rhombuses, etc. However, then he found that these could be subdivided into only two even more basic tiles (for instance, the ones that we used above – thin and thick rhombuses). From these basic tiles, he was able to prove their non-periodicity and relation to the Golden ratio.

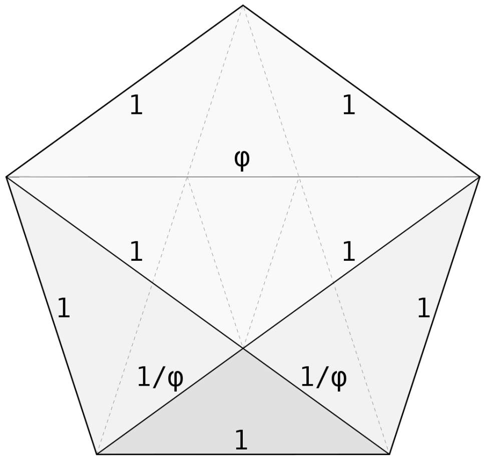

Figure 3.20: Regular pentagon with sides of length equal to one. Penrose analyzed such pentagons before he discovered the non-periodic set of basic tiles (Penrose, 1979a, 1979b). Note that in the picture they use φ instead of ϕ for the golden ratio, Source of the image: Wikipedia contributors. (2023). *Penrose tiling*. Retrieved 5/19/2023, from https://commons.wikimedia.org/w/index.php?curid=120315440.

The golden ratio can be defined as the ratio between the whole and part which is the same for all the scales (see Fig. 3.21), thus:

$$\phi = \frac{a+b}{a} = \frac{a}{b}$$

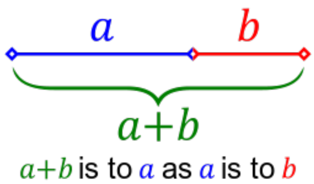

Figure 3.21: Golden ratio definition portrayal

Now, to see why this equation approaches the golden ratio, rearrange the equation to express b in terms of a and substitute the expression for b into the equation in the middle.

$$\phi = \frac{a+b}{a} = \frac{a+\frac{a}{\phi}}{a}$$

Simplify the equation and isolate ϕ:

$$\phi^2 = \phi + 1$$

Rearrange the equation to form a quadratic equation:

$$\phi^2 - \phi - 1 = 0$$

Use the quadratic formula to solve for ϕ:

$$\phi = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

Simplify the expression:

$$\phi = \frac{1 \pm \sqrt{5}}{2}$$

Choose the positive solution as the golden ratio:

$$\phi = \frac{1 + \sqrt{5}}{2}$$

The golden ratio famously occurs in ***Fibonacci sequence*** as a ratio between the neighboring numbers:

$$F_n = F_{n-1} + F_{n-2}$$

with $F_0 = 0, F_1 = 1$

The claim is that

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi$$

Call the ratio between consecutive numbers

$$R_n = \frac{F_{n+1}}{F_n}$$

Inductive step: Assume that ratios R_n and R_{n-1} converge to ϕ as $n \to \infty$. We need to show that R_{n+1} also converges to ϕ.

From the Fibonacci sequence:

$$F_{n+2} = F_{n+1} + F_n$$

Define $R_{n+1} = \frac{F_{n+2}}{F_{n+1}}$

$$R_{n+1} = \frac{F_{n+1} + F_n}{F_{n+1}}$$

Now, express R_{n+1} in terms of R_n:

$$R_{n+1} = \frac{F_{n+1}}{F_n} \cdot \frac{F_n}{F_{n+1}} + \frac{F_n}{F_{n+1}} = \frac{1}{R_n} + 1$$

Taking the limit as $n \to \infty$:

$$\lim_{n \to \infty} R_{n+1} = \lim_{n \to \infty} \left(\frac{1}{R_n} + 1 \right)$$

Since we assumed $R_n \to \phi$, as $n \to \infty$:

$$\lim_{n\to\infty} R_{n+1} = \frac{1}{\phi} + 1$$

From the definition of the golden ratio, we know that:

$$\frac{1}{\phi} + 1 = \phi$$

Therefore,

$$\lim_{n\to\infty} R_{n+1} = \phi$$

Thus, the ratio between neighboring Fibonacci numbers converges to the golden ratio as n $\to \infty$. □

This divine proportion along with Fibonacci numbers is approximately found in various objects in nature and art. Interestingly, the Golden ratio has been called the ***Divina proportione*** since the 15th century, when Pacioli delved into the Fibonacci sequence and found that this proportion is the ratio between the neighboring Fibonacci numbers. Then, Pacioli gave five reasons why the golden ratio should be referred to as the ***Divine Proportion***:

1. Its value represents divine simplicity since it can be so easily defined.
2. Its definition invokes three lengths, symbolizing the Holy Trinity.
3. Its irrationality (in the sense of distinction between rational and irrational numbers) represents God's incomprehensibility.
4. Its self-similarity recalls God's omnipresence and invariability.
5. Its relation to the dodecahedron, which represents the quintessence.

While it might sound somewhat mystical, it is a good indicator for mathematicians that something is true if it has aesthetical properties (Penrose, 1974), and the Golden section does have such properties.

Let us recap this chapter. From the formal point of view, the concept of perfect global symmetry is too restrictive and idealizing. Despite being useful, it cannot capture complex fractal objects satisfactorily. The concept of partial symmetries is less restrictive and has this potential since it allows minor flaws but still provides enough structure as opposed to even more abstract algebraic structures, and so it provides a good balance in-between too much order and too much chaos.

Moreover, we saw that the concept of partial symmetry captures also objects that have self-similar and proportional nature, therefore, it can be viewed as a formalization of the ancient concept of symmetry which was defined in terms of proportion.

Chapter 4

Philosophy

While the symmetry principle embraces the whole of nature from microworld to macrocosm, it fails, nevertheless, to explain one most important phenomenon-the phenomenon of growth, of change, which the principle of proportion implies
... symmetry is the statics of nature,
... proportion is its dynamics

Voloshinov

In this chapter, we delve into the philosophy of (partial) symmetry and synthesize the knowledge from the previous chapters. We argue that partial symmetries and their formal corresponding concept of inverse monoids are fundamental. The whole argument can be summarized as follows:

Argument 1: Global symmetry fails to describe even the slightest asymmetries of the world**.**

Premise 1: The concept of global symmetry cannot capture objects with just slight asymmetries. (restrictivity)

Premise 2: Objects of the world have slight asymmetries. (fundamental "imperfection" of the world)

Conclusion: The concept of global symmetry is too restrictive to fully capture the objects of the world.

Argument 2: Symmetry does not describe the complexity**.**

Premise 1: Global symmetries are characterized by repetitive and predictable information, resulting in informational redundancy; this allows one to infer the whole structure from just one part.

Premise 2: Complex systems exhibit unpredictable behavior, chaotic dynamics, high sensitivity to initial conditions, and fractal-like self-similarity across different scales.

Conclusion: Due to their inherent unpredictability, sensitivity to initial conditions, and scale-invariant self-similarity, complex systems' behavior cannot be fully captured by global symmetries, which rely on informational redundancy and predictability.

Argument 3: Perfect global Symmetry might be a cognitive simplification**.**

Premise 1: Our cognitive faculties tend to simplify complex information by seeking patterns, regularities, and symmetries in the world around us.

Premise 2: Perfect global symmetry represents a notion of perfect order, predictability, and simplicity, which may not necessarily exist in the intricate and often chaotic nature of reality.

Premise 3: The concept of perfect global symmetry, though mathematically and aesthetically appealing, may serve as a mental construct or idealization that helps us comprehend and make sense of the complexity of the world.

Conclusion: The idea of perfect global symmetry might be an idealization or illusion of our minds, stemming from our cognitive tendency to simplify complex information and seek patterns, rather than reflecting an inherent property of the real world.

Argument 4: Local (Partial) Symmetry is a potential candidate**.**

Premise 1: Complex systems are characterized by inherent unpredictability and scale-invariant self-similarity, which result in intricate patterns and structures.

Premise 2: Local (partial) symmetry offers a more nuanced perspective by integrating unpredictability, underlying structures, and proportion (self-similarity) in addition to being a generalization of perfect global symmetry of group theory.

Conclusion: Given its ability to account for the diverse characteristics of complex systems, including unpredictability, scale-invariant self-similarity, and underlying structures, local (partial) symmetry emerges as a promising framework for understanding and capturing the behavior of such systems.

Argument 5: Local (Partial) Symmetry is a useful candidate**.**

Premise 1: Group theory can be studied using permutations, which make the theory useful.

Premise 2: More general structures, such as bare monoids and semigroups, tend to lack the structure that would be graspable using useful combinatorial structures.

Premise 3: Inverse monoids can be studied using partial permutations.

Conclusion: Therefore, inverse monoids are preferable to more general structures. (they are rich enough, but they still have a lot of structure – that is because of the nature of the inverse property, which is a generalization of the inverse property of group theory)

Many of these premises and conclusions were already reached in previous chapters. However, we will elaborate on these arguments in the following sections.

Moreover, we consider what kind of fundamentality is to be attributed to this concept. Does it have any ontological or epistemological commitments?

By examining these arguments and questions, we aim to elucidate the profound importance of partial symmetry in understanding the complex fabric of our reality.

Finally, we provide some implications if the argument holds.

4.1 The limits of Symmetry

In this section, let us deal with the Argument 1. Firstly, let us focus on the first premise of the argument, i.e., restrictivity of the concept of global symmetry. In Chapter 3 on mathematics, we contrasted the concept of symmetry with the concept of partial symmetry, especially on graphs. The result was that the concept of global symmetry could not capture the objects with just a slight asymmetry, such as the minimally asymmetric graphs in Fig. 3.9 which, despite their non-existent global asymmetry, have all underlying induced subgraphs globally symmetric.

Furthermore, Senechal (1989) explores the limitations of group theory in understanding certain aspects that one would like to be captured in the concept of symmetry. Classical group theory focuses on global properties rather than local configurations (for details, see Section 3.2). Senechal (1989) points out that "many of the most interesting problems in contemporary symmetry theory concern local configurations, and group theory may not be the only or the best tool for studying them". This is because "the group of automorphisms can give us information about the structure as a whole - that is, it characterizes its global properties - but it does not always help us to understand the reasons why these properties exist". As we know from the world around us, objects tend to be organized from the bottom-up into the structures, Senechal claims that this bottom-up approach is non-existent in the concept of perfect global symmetry. Although, the concept of perfect global symmetry is frequently complemented with symmetry breaking to explain the bottom-up approach, it still lacks the explanation of why should the structures organize to global symmetries at the next level.

Secondly, let us comment on the second premise of the argument, i.e., fundamental imperfections of the world. It is obvious that observable objects in the world have imperfections. Despite this, there is a common scientific narrative, that there are underlying symmetries at the deepest level (supersymmetry of theory of everything). But, such a level has yet not been found. Moreover, modern physics describes and unifies particles as certain types of groups, but as is well known to physicists, such unification are only approximations with perfect symmetry (Mainzer, 1996). What we currently empirically observe is that the objects of the world have at least slight asymmetries that at best can be approximated by perfect symmetries.

Icosahedral quasicrystals, which have gained attention due to their five-fold symmetry, are another example where group theory is insufficient for understanding their structure. Instead, it is suggested that these quasicrystals may be a three-dimensional analog of Penrose tiles, see for instance Fig. 3.14. In this context, local matching rules are more relevant than the global perspective of the group theory (Senechal, 1989).

Thus, I think that the conclusion that the concept of global symmetry cannot capture the objects of the world makes sense. While I agree that the concept of global symmetry has ever had a great unificatory, descriptive, and heuristic power in the search for ultimate laws, it turns out it cannot deal structurally with complexity.

4.2 (Partial) symmetry and complexity

In this section, let us deal with the Argument 2, i.e., the fact that perfect global symmetry cannot capture the complexity, and Argument 4, i.e., the possibility of partial symmetry being a possible candidate in its stead.

It is clear from the observations of the world that there are chaotic fractal structures which are unpredictable due to inherent high sensitivity to the initial conditions. Many of these have unpredictable and self-similar behavior at several scales.

As we showed in Chapter 3, the concept of perfect global symmetry cannot capture self-similarity even in the trivial case of the Sierpinski triangle in Fig. 3.13 since it represents all the steps of its generation in as one and the same group with six transformations, completely ignoring the rich underlying structure.

However, structures such as the one in the Sierpinski triangle can perhaps be adequately represented using inverse monoids which are considered as formalization of local or partial symmetries (Lawson, 1998).

For the most part, complexity, fractality, and self-similarity require some sort of unexpectedness, change, and phase transitions. While global symmetric structures can in some cases provide the phase transitions such as in the case of repeating the same crystalline structures until some transition happens due to immense inner pressure (symmetry breaking), they do not structurally describe the self-similarity across various scales.

It is a joint strife of modern science to look for ever deeper underlying symmetries. But "while the symmetry principle embraces the whole of nature from microworld to macrocosm, it fails, nevertheless, to explain one most important phenomenon-the phenomenon of growth, of change, which the principle of proportion implies" (Voloshinov, 1996). Since symmetrical objects are highly information-redundant objects, without some sort of imbalance, it seems impossible to generate complexity.

Before the 15th century, see Section 0.1, Aristotle, Plato, Pythagoreans, Vitruvius, and many other thinkers associated the concept of symmetry with proportion. Obviously, their usage was not well formalized. However, in our current state of knowledge, we might have found such a formalization in terms of inverse monoids. This formalization allows us to consider proportionality that is not covered under the modern concept of symmetry.

Voloshinov (1996) acknowledges that there might be a complementary principle to the principle of symmetry that would account for the "phenomenon of growth". He distinguishes between symmetry and proportion, stating that "symmetry is the equality of states, the invariance of state-balance," and "proportion is the equality of changes, the invariance of change-imbalance". This distinction reflects the statics and dynamics of nature, where "symmetry is the statics of nature," and "proportion is its dynamics".

However, he further describes the relationship between fractal geometry and chaos theory and claims that "the relationship of chaotic and fractal forms is also conditioned by the symmetry principle, for the invariance of time changes in chaotic phenomena is analogous with space changes in fractal forms" (Voloshinov, 1996). I am not sure whether he meant that the other principle is reducible to the former one. If that is so, I find it unconvincing. While I agree that the principle of symmetry might condition even the fractal forms due to global invariance of time, nevertheless it does not mean that the fractality is reducible to global symmetry. In fact, the Sierpinski triangle is a very simple counterexample.

Interestingly, the golden ratio, a unique geometrical proportion, exhibits a special kind of symmetry. Voloshinov (1996) states that "similarity of form is achieved in any geometrical proportion, but it is only in the geometrical proportion of the golden section that the similarity of form is achieved within the framework of the whole. Thus, the golden section is a symmetry of similarity of the parts and the whole", see Section 3.4.2 for more details on the golden ratio, where we demonstrated that the golden section is a perfection of self-similarity and occurs in complex objects such as in tilings that are not described well by the concept of symmetry.

Partial symmetry can be viewed as a property that can help us describe structures in the spectrum from order and chaos, as illustrated in Fig. 4.1. It is a concept that deals with partial permutations and partial permutations are in each of these structures, thus it could be an overarching concept that can capture the structure (if they have some) of all of these objects. Especially interesting ones are those that are commonly generated by human behavior, such as networks of power grids, or found in living systems such as neural networks. These structures tend to have scale-free fractal-like structures. Similarly, Wolfram (1984) classified cellular automatons into four classes based on their behavior, see Fig. 4.2. The cellular automaton is a formal system that evolves over time, with each row in the matrix representing a single time step. Every cell can be either black or white. At each time step, the value of each site in the consecutive is updated using cellular automaton rules based on the cell's neighborhood. Class 1 cellular automata result in limit sets containing only periodic configurations, similar to limit cycles or limit points in dynamical systems. Class 2 cellular automata result in chaotic and aperiodic limit sets, which contain analogues of chaotic or 'strange' attractors. Class 3 automata result in irregular changes and behavior that is effectively unpredictable but with 'life-like' structures. Class 4 automata results in noisy random behavior. These rules take into account the values of neighboring sites from the previous time step.

Aperiodic quasicrystal-like graph   Random graph

Grid graph   Graph with scale-free distribution

Figure 4.1: From the highly ordered graph to the highly chaotic random graph.

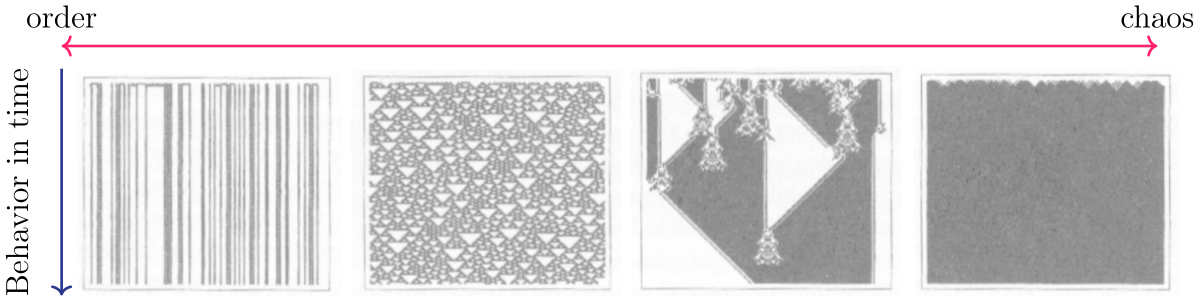

Figure 4.2: Instantiations of the behavior of four different classes of cellular automatons based on their behavior. After Mainzer (2005).

4.3 Symmetry and Cognition

In this section, let us comment on the Argument 3, i.e., the fact that people naturally tend to look for simplicity and perfection, which might suggest that the concept of perfect symmetry is an idealization.

4.3.1 Symmetry arguments and symmetry principles

Brading et al. (2003) distinguish between ***symmetry principles*** and ***symmetry arguments*** in the philosophical jargon. The need for this distinction relates to its usage. Symmetry principles are symmetry properties attributed to laws, and it is this application that "has become central to modern physics" (Brading et al., 2003). On the other hand, symmetric properties attributed to phenomena and objects are called symmetry arguments.

Recall from the Section 0.1 that among others, Aristotle, Plato, Galileo, Kepler, and Leibniz were intuitively using symmetry principles. Aristotle principally assumed that the world is made out of substances. He used the concept of ***substance*** to signify that which subsists (stays invariant) in time, and God for him was a substance that is the most perfect, and the most simple – which he imagined as a circle because the circle is perfectly self-sufficient, i.e., it begins and ends in itself (Aristotle, 2016; Mainzer, 1996). Or, Leibniz used the principle of sufficient reason, which roughly says that "if there is no sufficient reason for one thing to happen instead of another, then nothing happens (the initial situation does not change)"(Brading et al., 2003). As an illustration, take the problem of Buridan's ass: imagine an ass who is standing in front of two completely equivalent stacks of hay, he would have no reason to choose since both stacks are equivalent, and so he would starve to death. In other words, "a situation with a certain symmetry evolves in such a way that, in the absence of an asymmetric cause, the initial symmetry is preserved... a breaking of the initial symmetry cannot happen without a reason, or an asymmetry cannot originate spontaneously" (Brading et al., 2003).

Moreover, one can experience and expect properties like symmetry (a type of invariance transformation) without explicit knowledge of it. These considerations suggest that symmetry principles and arguments can be easily seen as a manifestation of our mind's propensity to simplify the world by seeking and expecting simplicity in the world. We reviewed these tendencies in Chapter 2 and Chapter 1. These cognitive tendencies may explain why we are drawn to symmetries in various domains, even though they might not always accurately describe the underlying reality.

Furthermore, Sarukkai (2004) argues that Gestalt principles of perceptual organization are strikingly similar to the axioms and ideas of group theory, his views can be summarized in the Table 4.1. Based on his analysis and comparison, Sarukkai proposes the question for further research: "Why do we model even non-visual [mathematical] symmetries in the same way as we do the visual ones, especially since the visual form is constructed via the Gestalt?"

Sarukkai speculates that both might be reducible to the principles of perception. Accumulated evidence, that we provided in this thesis, supports this speculation.

Gestalt Psychology	**Mathematical Group theory**
Perception of whole structures, e.g., seeing a triangle formed by three dots, instead of seeing the sum of the parts.	The concept of a group is also analogous to seeing a whole structure (a closed set of transformations that reorder the parts), e.g., a triangle from discrete parts (three dots).
The closure property in Gestalt principles suggests that perception fills in the gaps, projecting a complete form even if some elements are missing.	Closure axiom ensures that the operation between any two elements in the set produces another element within the same set. Thus, elements of a group can be filled in the gaps, if sufficient information is available to infer it from.
'Identity' is akin to the 'first impression' in Gestalt psychology.	Identity element signifies the element that leaves others unchanged when operated with.
The laws of organization include principles of similarity and symmetry, an instinctual recognition of patterns and shared characteristics.	The group structure captures the idea of an inverse for each element. Moreover, group structure captures the idea of global symmetries.

Table 4.1: Comparison between Gestalt principles of perceptual organization and mathematical group theory.

4.3.2 What if there are no laws, just symmetries?

On the other hand, Rosen (1989) demonstrated that the very concepts of what we understand by ***lawfulness*** and ***objectivity*** of science in the sense of both reproducibility and predictability rests on the notion of perfect symmetry.

Reproducibility is commonly defined as "the same experiment always gives the same result" (Rosen, 1989). Obviously, each experiment is unique due to differences in time, location, and other factors, so the term "same" should be understood as 'equivalent under a set of transformations of factors' rather than 'identical'. Identical experiments are impossible due to differences in time, location, and other factors, and instead he proposes the concept of equivalent experiments that can be transformed into one another through a set of defined transformations. Reproducibility is then defined as a relation that holds for all transformations within a set of possible experiments, ensuring that the transformed result is what is obtained when performing the transformed experiment (Rosen, 1989).

Predictability is the ability to predict the results of new experiments using a reproducibility relation R that connects experiments and their results: $result = R(experiment)$. Obviously, this relation is symmetrical, because "the experiment and its result obtained by changing the experimental input obey the same relation as the original experiment and result" (Rosen, 1989). Consider all the different experiment-result pairs (exp, res) that have been, will be or could be obtained by doing the experiment. You can simply transform any one of these into another by replacing it. The transformed pair is different from the original one, but the pairs have the same thing in common that the transformation does not change. This feature is that result and experiment always follow the relationship $result = R(experiment)$ for all possible couples $(experiment, result)$.

Moreover, Noether (1971) showed that if a system has a continuous symmetry property, then there are corresponding quantities of the system whose values are conserved in its processes (conservation laws), thus conservation laws can be causally reduced to considerations of symmetries (Brading et al., 2003; Sarukkai, 2004). For instance, if the systems are claimed to be explicitly independent/invariant under transformations of time, the systems will conserve the total energy (Sarukkai, 2004).

Similarly, Van Fraassen (1989) conceives of the ***Symmetry Requirement***, an imperative principle anchoring the process of generating symmetry arguments. This principle posits that "problems which are essentially the same must have essentially the same solution", thereby emphasizing the element of equivalence and analogue in problem-solving.

Van Fraassen, in agreement with Rosen, asserts that the symmetries of time, space, and motion are instrumental in shaping the architecture of modern science, with implications for theory development. We demonstrated this with Rosen's considerations of reproducibility and predictability. Thus, symmetry, as a conceptual tool, rigorously influences the formulation and articulation of all scientific theories to some extent.

An archetypal instance of a 'pure' symmetry argument, as per van Fraassen, can be found in the realm of probability theory, which has, since its advent in the seventeenth century, often been guided by the belief that symmetry can determine probability. This belief finds expression in ***equipossibility*** leading to equal probability and is acknowledged in terms like indifference and sufficient reason. In other words, without any prior knowledge, we assume uniform distribution.

Moreover, would we be able to understand what is the concept of partial local symmetry without the concept of perfect global symmetry? We can only know that something is not perfect by first having the idea of perfection.

Based on these considerations, I admit that the narrative of perfect symmetry principles might not be entirely avoidable, at least not at the fundamental level. However, these principles can be subsumed under the broader and more inclusive concept of partial symmetry as the concept of global symmetry is just a subconcept of partial symmetry, we proved this in Section 3.3. By recognizing partial symmetry as a generalization of the traditional concept of symmetry, moreover, we can accommodate a wider range of structural features and relationships found in complex systems that seem not to be possible to accommodate within the global perfect symmetry without relying on the narrative of symmetry breaking. This generalized framework allows for a more nuanced understanding of symmetry and its role in scientific theories, while simultaneously preserving the essential insights that global symmetries provide.

4.4 Partial symmetry and structure

In this section, let us expand on the premises and conclusion of Argument 5. The question under consideration is roughly this – why not even more general algebraic structures such as semigroups, or monoids without the inverse property?

Firstly, consider that only inverse monoids still have some non-trivial structure, which is available because of the inverse property. This inverse property of inverse monoids is a generalization of the inverse property of the group theory. Without the inverse property, algebraic structures lack the essence of what symmetry means.

Moreover, group theory can be studied using permutations, which has led to numerous applications and insights in various areas of mathematics, physics, and other disciplines. The availability of representation theorems in group theory allows for connections between abstract algebraic structures and more concrete, geometric, or linear-algebraic entities, thus facilitating combinatorial approaches to problem-solving.

More general structures such as bare semigroups and monoids lack the combinatorial tools and structures that make group theory so powerful. This limitation restricts the applicability of these more abstract structures in various contexts.

Inverse monoids, as a generalization of groups, possess representation theorems that enable similar combinatorial approaches to those found in group theory via partial permutations and partial automorphisms. This parallelism allows for the development of a rich theory surrounding inverse monoids, making them a suitable alternative to global symmetries in group theory.

Thus, given their combinatorial advantages, inverse monoids offer a more versatile and powerful alternative to more abstract algebraic structures such as semigroups and monoids.

4.4.1 Structuralism

In Chapter 3, we dealt with the formalism of the concept of partial symmetry in terms of inverse monoids, partial permutations, and partial automorphisms of graphs. From a philosophical point of view, we need to ask whether the concept of partial symmetry involves any ontological and paradigmatic commitments.

Firstly, let us make it clear – it is a common view, and I completely agree that group theory, inverse monoid theory, and in general any mathematical theory is not inherently aligned with any philosophical stance or worldview (Einstein, 1921; Poincaré, 1952). Mathematical theories deal with mathematical objects and structures, such as sets of elements that obey certain axioms and properties. As such, mathematical theories are neutral with respect to philosophical debates and are purely concerned with formal and logical relationships within these structures, which are defined axiomatically.

However, the applications of the theories, and motivation for their development, can be influenced by various philosophical perspectives.

Let me now comment on paradigms that are usually employed in applications of these theories. By ***paradigm***, I understand "the set of common beliefs and agreements shared between scientists about how problems should be understood and addressed" (Kuhn, 2012). It is a "way of seeing", a perspective (a set of beliefs and assumptions) one adopts when dealing with the world.

Then, ***structural realism*** is a scientific paradigm in that scientific theories describe the underlying structure of the world, rather than the world itself. It is a form of 'post-Kantian' scientific realism that emphasizes the importance of the structure of scientific theories as opposed to entities. More specifically, these are some common assumptions of structural realism:

1. The world exists.

2. The world has a stable objective structure (and only to this structure do we have access).
3. The entities that make up the structure of the world are (almost) irrelevant (they cannot be perceived directly in themselves).
4. What matters are the patterns of relationships between these entities (our perception of the world preserves the structure, but not necessarily the entities).
5. The structure of the world is hidden.
6. Scientific theories aim to uncover the structure of the world.
7. The structure of the world is the basis for scientific progress.

For a more detailed description, see for instance Ladyman (1998) or a critique, and a semantic alternative of this approach in Van Fraassen (1989). Group theory and inverse monoid theory may subscribe in their applications to structural realism for several reasons:

1. These theories provide a formal framework to study structures that can be known and understood independently of the specific entities they describe. An important concept is the notion of invariant features, which remain unchanged under specific transformations. This aligns well with structural realism's focus on the structures.
2. They have been successfully applied to various branches of natural sciences, such as physics and chemistry, where they describe both the structures in which the entities are involved and the entities themselves. For instance, in particle physics, particles are considered just as instantiations of certain groups, thus reducing entities to mere structures (Mainzer, 1996).
3. Both theories can be used to unify seemingly disparate mathematical structures and scientific theories. The ability to identify common structural features across different domains supports the idea that structure, rather than entities, is the essential aspect of our knowledge of the world.

Furthermore, since these theories are employed in various empirical domains such as physics and chemistry, obviously there are overlaps with the paradigm of ***positivism***, especially in their commitment to verificationism, reductionism, and quantificationism as specified for instance by Popper (2005). Moreover, the whole argument of this thesis itself relied several times on demonstrations in experiments.

But, the concept of symmetry is not confined to the objective physical world which was the sole focus of positivists. Instead, this concept can also be extended into the subjective realm, encompassing psychology, aesthetics, and even motivations in philosophy and science. This dual role is intriguing and suggests the deep and fundamental nature of symmetry in our understanding of reality, crossing these arbitrary boundaries of subjectivity and objectivity.

There are also evident commitments in applications to ***complexity*** perspective, which might be succinctly summarized as follows:

1. The world is complex.

2. Chaos, non-linearity, self-organization, self-similarity, emergence and phase transitions, interactions, and dynamics are concepts that capture what complexity is about (Heylighen et al., 2006).

The concept of partial symmetry is particularly relevant to complex structures, as it allows us to capture the underlying patterns and order in these systems without requiring complete symmetry or perfect order. This is important because many complex systems exhibit only partial symmetry, due to the presence of factors such as random fluctuations, noise, or other sources of variability. Therefore, it is important to recognize that the concept of global symmetry, while useful in many contexts, can be too restrictive and idealizing when it comes to generating complex structures. By embracing the complexity paradigm towards science, we can move beyond the limitations of perfect global symmetry and explore the rich and diverse range of patterns and structures that emerge from complex systems.

Conclusion

The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.

G.H. Hardy

This thesis has explored the concept of partial symmetry. By examining its evolution, its aspects in arts, nature, and cognition, along with instances of symmetry breaking in physics and its philosophical implications, we have argued that the observed lack of perfect symmetry might be an essential aspect of the natural world and that perhaps the concept of partial symmetry might be a suitable candidate in its stead for the description of the structures. This can be viewed as a sort of return to the original meaning of the word but with precise mathematical formulation. Moreover, it captures what master tilers were doing when they tiled a plane. Our thesis challenges conventional perspective and opens new avenues for inquiry in both science and philosophy.

If the argument presented holds, then this has several implications for our understanding of nature in general.

Firstly, it would be beneficial to re-evaluate symmetry principles, perhaps looking for partial symmetry principles, potentially leading to novel theories and frameworks that better account for the observed asymmetry in natural systems.

From a philosophical standpoint, the absence of perfect symmetry could change our perspective on the nature of reality, prompting us to reconsider the relationship between idealized concepts and the inherently imperfect world we inhabit.

In future research, it might be useful to analyze some of the key terms that are associated with local symmetry such as causality, invariance, transformation, symmetry principles, proportionality, laws, etc. Moreover, it would be beneficial to further contrast the two narratives – one that we told about partial symmetry, and the other which relies on the concept of symmetry and symmetry breaking. I think that both can be complementary. Symmetry and symmetry breaking already serve as useful guides in describing the process of generating complexity. These two concepts taken together provide procedural explanations. In contrast, partial symmetries provide a structural explanation of the current state that was generated by the process of symmetry breaking.

This thesis has at least these limitations and weak points that I am aware of:

1. Since this thesis is a synthetic work, in many ways, my thesis will just scratch the surface of the ongoing research on the concept of symmetries, and therefore I might miss a lot of information that would stand against this argument.

2. I do not discard the possibility of me unintentionally looking for information that would strengthen this argument in the synthesis.

3. Several times in the thesis, I had to rely on analogies and the intuition of the reader instead of writing all the details, because of lack of time, space, and comprehensibility for a wider audience, and since the formal area of partial symmetry in mathematics is still an ongoing research.

4. Moreover, while I tried to meaningfully combine information from various sciences, and I am not an expert in any of these fields, thus, I mostly rely on their methodologies and sense-making. This might at times have led to an illusion of understanding on my side, and unintentional misinterpretation that I might have included in the thesis.

Nevertheless, to the best of my knowledge, this thesis is the first modern interdisciplinary and systematic (and hopefully consistent) exploration of the concept of partial symmetries. Partial symmetries are a multifaceted topic with great relevance in both art and nature. Its foundation lies in mathematics, similar to global symmetries. My aim was to argue that this concept is in some sense better (and more beautiful) than the concept of global symmetry, provide you with an overview of its numerous aspects, and guide you from intuitive concepts to more complex, abstract ideas. I hope I have succeeded in this endeavor.

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